

$$\begin{aligned} &\leq |u(0)|^p + \frac{n^{p+1}}{2^p} D_{p,n} \int_0^1 \int_S |\nabla u(\rho\zeta)|^p d\sigma_N(\zeta) (1-\rho^2)^{p-1} \rho^{n-1} d\rho \\ &= |u(0)|^p + \frac{n^p}{2^p} D_{p,n} \int_{\mathbb{B}} |\nabla u(x)|^p (1-|x|^2)^{p-1} dV_N(x), \end{aligned}$$

as desired.

**Remark 2.** Note that when  $p \leq 1$  and  $u$  is a positive harmonic function, then by Jensen's inequality and the mean value property we have that

$$\left( \int_S u^p(r\zeta) d\sigma_N(\zeta) \right)^{1/p} \leq \int_S u(r\zeta) d\sigma_N(\zeta) = u(0),$$

that is,  $u^p$  is superharmonic. From this and by Fatou lemma it follows that  $\int_S [u^*(\zeta)]^p d\sigma_N(\zeta) \leq \|u^*\|_{\mathcal{H}^p}^p \leq u(0)^p$  whenever  $u$  is a positive harmonic function.

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## ON THE AUTOMORPHISM GROUP OF A POSSIBLE SYMMETRIC (81, 16, 3) DESIGN

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ABSTRACT. In this paper we study the automorphism group of a possible symmetric (81, 16, 3) design.

### 1. INTRODUCTION

Let  $v$ ,  $k$  and  $\lambda$  be non-negative integers such that  $v > k > \lambda$ . By a symmetric  $(v, k, \lambda)$  design, we mean a pair  $D = (V, \mathcal{B})$ , where  $V$  is a  $v$ -set and  $\mathcal{B}$  is a set of  $k$ -subsets of  $V$  such that the following four requirements are satisfied by  $D$ :

- (1)  $|\mathcal{B}| = v$ .
- (2) any element of  $V$  belongs to precisely  $k$  members of  $\mathcal{B}$ .
- (3) any two distinct members of  $\mathcal{B}$  intersect in exactly  $\lambda$  elements of  $V$ .
- (4) any two distinct elements of  $V$  are in exactly  $\lambda$  members of  $\mathcal{B}$ .

As usual, the elements of  $V$  are called points of  $D$  and the members of  $\mathcal{B}$  are called blocks of the design  $D$ . An automorphism of a symmetric design  $D = (V, \mathcal{B})$  is a permutation on  $V$  which sends blocks to blocks. The set of all automorphisms of  $D$  with the composition rule of maps forms the full automorphism group of  $D$  which will be denoted by  $Aut(D)$ . If  $\alpha$  is an automorphism

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of  $D$ , we denote by  $F(\alpha)$  the set of all points which are fixed by  $\alpha$ ; and  $F_b(\alpha)$  denotes the set of all blocks which are fixed by  $\alpha$ .

Over the years, researchers have tackled problems related to symmetric designs. The question of existence still remains unsettled for many parameter sets. Indeed, if we list the parameters  $(v, k, \lambda)$  in order of increasing  $n = k - \lambda$ , then  $(81, 16, 3)$  would be the smallest unknown case [8]. On the other hand, the success of almost all the design construction methods depends heavily on a proper choice of possible automorphism groups [4].

As far as we know, the only known results on a possible  $(81, 16, 3)$  design are the following:

**Theorem 1.1.** (See [2]) *There is no symmetric  $(81, 16, 3)$  design with an abelian regular 3-group of automorphisms.*

**Theorem 1.2.** (See [7]) *Let  $\alpha$  be an automorphism of a possible symmetric  $(81, 16, 3)$  design of order 2. Then  $|F(\alpha)| = 9$ .*

**Theorem 1.3.** (See [5]) *The alternating group  $A_5$  of degree 5 cannot be isomorphic to a group of automorphisms of a possible symmetric  $(81, 16, 3)$  design.*

T. Spence has announced in his home page

<http://www.maths.gla.ac.uk/~es/>

that there is no symmetric  $(81, 16, 3)$  designs having a "certain" fixed-point free automorphism of order 3.

Our main result is:

**Theorem 1.4.** *If  $G$  is the full automorphism group of a possible symmetric  $(81, 16, 3)$  design, then  $|G| = 2^\alpha 3^\beta 5^\gamma 13^\sigma$ , where  $\gamma \leq 1$ ,  $\sigma \leq 1$ . Moreover,  $G$  has no subgroup of order 65, and has no elements of orders 10 or 26; and  $G$  does not contain any abelian 2-subgroup of rank greater than 3.*

In Section 2, some general results on the automorphism groups of a symmetric design are given and in Section 3, we prove a series of Lemmata. Based on them we can prove Theorem 1.4.

## 2. SOME GENERAL RESULTS ON THE AUTOMORPHISM GROUP OF A SYMMETRIC DESIGN

**Lemma 2.1.** (See [6]) *Let  $\alpha$  be an automorphism of a nontrivial symmetric  $(v, k, \lambda)$  design. Then  $|F(\alpha)| = |F_b(\alpha)|$ .*

**Lemma 2.2.** (See [6, Corollary 3.7, p. 82]) *Let  $D$  be a non trivial symmetric  $(v, k, \lambda)$  design and  $\alpha$  a non trivial automorphism of  $D$ . Then  $|F(\alpha)| \leq k + \sqrt{k - \lambda}$ .*

**Lemma 2.3.** *Let  $D$  be a symmetric  $(v, k, \lambda)$  design and  $\alpha$  an automorphism of  $D$  of prime order  $p$  such that  $\lambda < p$ . If  $B$  is a block of  $D$  such that  $|F(\alpha) \cap B| \geq 2$ , then  $B^\alpha = B$ .*

*Proof.* Let  $x, y$  be two distinct elements of  $F(\alpha) \cap B$ . Then  $x, y \in B = B^{\alpha^0}, B^\alpha, \dots, B^{\alpha^{\lambda}}$ . Since every two distinct points are in exactly  $\lambda$  blocks,  $B^{\alpha^i} = B^{\alpha^j}$  for some distinct  $i, j \in \{0, 1, \dots, \lambda\}$ . Thus  $B^{\alpha^{i-j}} = B$ . Since  $p$  is prime and  $1 \leq |i - j| \leq \lambda < p$ ,  $\gcd(i - j, p) = 1$ . Therefore  $B^\alpha = B$  as required.  $\square$

**Lemma 2.4.** *Let  $B_1$  and  $B_2$  be two distinct fixed blocks of the automorphism  $\alpha$  of prime order  $p$  of a symmetric  $(v, k, \lambda)$  design with  $\lambda < p$ . Then  $B_1 \cap B_2 \subseteq F(\alpha)$ .*

*Proof.* Suppose, for a contradiction, that there exists a point  $x \in (B_1 \cap B_2) \setminus F(\alpha)$ . Thus  $x^{\alpha^i} \neq x^{\alpha^j}$ , for any two distinct  $i, j \in \{0, 1, \dots, p - 1\}$ ; since otherwise  $x^{\alpha^{i-j}} = x$  and so  $x^\alpha = x$ , as  $\gcd(i - j, p) = 1$ . It follows that  $p = |\{x^\beta \mid \beta \in \langle \alpha \rangle\}|$ . Since  $B_i^\alpha = B_i$  for  $i \in \{1, 2\}$ , we have that  $\{x^\beta \mid \beta \in \langle \alpha \rangle\} \subseteq B_1 \cap B_2$ . Therefore  $|B_1 \cap B_2| \geq p > \lambda$ , a contradiction; since in symmetric  $(v, k, \lambda)$  designs, two distinct blocks intersect in exactly  $\lambda$  points.  $\square$

**Lemma 2.5.** *Let  $\alpha$  be an automorphism of prime order  $p$  of a symmetric  $(v, k, \lambda)$  design with  $\lambda < p$ . Then*

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \setminus F(\alpha)| \leq v.$$

*Proof.* It follows from Lemma 2.4 that for any two distinct blocks  $B_1$  and  $B_2$  in  $F_b(\alpha)$ ,  $(B_1 \setminus F(\alpha)) \cap (B_2 \setminus F(\alpha)) = \emptyset$ . This completes the proof.  $\square$

**Lemma 2.6.** *Let  $\alpha$  be an automorphism of a symmetric  $(v, k, \lambda)$  design of prime order  $p$  such that  $1 < \lambda < p$ . Then  $B \not\subseteq F(\alpha)$  for all blocks  $B$ .*

*Proof.* Suppose, for a contradiction, that there exists a block  $B$  such that  $B \subseteq F(\alpha)$ . Since every block  $B_1 \neq B$  intersects  $B$  in  $\lambda \geq 2$  points, it follows from Lemma 2.3 that every block is fixed under  $\alpha$ . Thus  $|F_b(\alpha)| = |F(\alpha)| = v$ , by Lemma 2.1. Hence  $\alpha$  is the identity automorphism; a contradiction. This completes the proof.  $\square$

The following lemma is Theorem 2.7 of Aschbacher's paper [1].

**Lemma 2.7.** (Theorem 2.7 of [1]) *Let  $p$  be a prime divisor of the automorphism group of a symmetric  $(v, k, \lambda)$  design such that  $1 < \lambda < p$  and  $\gcd(p, v) = 1$ . Then  $p \leq k$ .*

*Proof.* Suppose that  $\alpha$  is an automorphism of the design of order  $p$ . Since  $\alpha$  is a permutation on the point set,  $F(\alpha) \equiv v \pmod{p}$  and since  $\gcd(p, v) = 1$ , we have that  $|F(\alpha)| \geq 1$ . Thus, by Lemma 2.1, there exists a block  $B$  such that  $B^\alpha = B$ . Thus by Lemma 2.6, there exists an element  $x \in B \setminus F(\alpha)$  and so  $|\{x^\beta \mid \beta \in \langle \alpha \rangle\}| = p$ . Since  $B^\alpha = B$ , we have that  $\{x^\beta \mid \beta \in \langle \alpha \rangle\} \subseteq B$  and so  $p \leq k$ , as required.  $\square$

### 3. AUTOMORPHISM GROUP OF A POSSIBLE SYMMETRIC (81,16,3) DESIGN

**Lemma 3.1.** *Let  $G$  be an automorphism group of a possible symmetric  $(81, 16, 3)$  design which is elementary abelian 2-group. Then  $|G| \leq 8$ .*

*Proof.* Let  $r$  be the number of orbits of the action of  $G$  on the point set of the design. Then by the Cauchy-Frobenius Lemma (see [6, Proposition A.2, p. 246]),

$$r = \frac{1}{|G|} \sum_{\alpha \in G} |F(\alpha)|.$$

Since  $G$  is an elementary abelian 2-group, it follows from Theorem 1.2, that  $|F(\alpha)| = 9$  for all non-identity elements  $\alpha$  of  $G$ .

Let  $|G| = 2^n$ . Then, since  $r = (2^n + 8) \cdot 9/2^n$  is an integer, we must have that  $2^n$  divides  $2^n + 8$  and so  $n \leq 3$ , as required.  $\square$

**Lemma 3.2.** *Let  $G$  be an automorphism group of a possible symmetric  $(81, 16, 3)$  design. Then  $G$  has no element of order 7 or 11.*

*Proof.* Suppose, for a contradiction, that  $G$  has an automorphism  $\alpha$  of order  $p$ , where  $p \in \{7, 11\}$ . Since  $\alpha$  is a permutation on a set with 81 elements, we have  $|F(\alpha)| \equiv 81 \pmod{p}$ . Then it follows from Lemma 2.2 that

$$|F(\alpha)| \in \begin{cases} \{4, 11, 18\} & \text{if } p = 7 \\ \{4, 15\} & \text{if } p = 11 \end{cases} \quad (I)$$

Thus there are at least two distinct blocks which are fixed by  $\alpha$  and so

$$|F(\alpha)| \geq 3 \quad (*)$$

by Lemma 2.4. Now if  $B \in F_b(\alpha)$ , then  $\alpha$  induces a permutation on the set  $B$ . Therefore  $|F(\alpha) \cap B| \equiv 16 \pmod{p}$  and so by  $(*)$  and Lemma 2.6 we have

$$|F(\alpha) \cap B| = \begin{cases} 9 & \text{if } p = 7 \\ 5 & \text{if } p = 11 \end{cases} \quad (II)$$

If  $p = 11$ , then it follows from  $(I)$  and  $(II)$  that  $|F(\alpha)| = 15$  and  $|B \setminus F(\alpha)| = 11$  for all blocks  $B \in F_b(\alpha)$ ; and if  $p = 7$ , then  $|B \setminus F(\alpha)| = 7$  for all blocks  $B \in F_b(\alpha)$  and  $|F(\alpha)| \in \{11, 18\}$ . Both cases contradict Lemma 2.5. This completes the proof.  $\square$

**Lemma 3.3.** *Let  $\alpha$  be an automorphism of a possible symmetric  $(81, 16, 3)$  design of order 5. Then  $|F(\alpha)| = 1$ .*

*Proof.* Since  $\alpha$  is a permutation on the point set, it follows from Lemma 2.2 that  $|F(\alpha)| \in \{1, 6, 11, 16\}$ . Suppose, for a contradiction, that  $|F(\alpha)| \neq 1$ . Let  $B = B_1$  be an arbitrary block in  $F_b(\alpha)$ . Since  $|F_b(\alpha)| = |F(\alpha)| \geq 2$ , there exists a block  $B_2 \neq B_1$  in  $F_b(\alpha)$ . By Lemma 2.4,  $B_1 \cap B_2 \subseteq F(\alpha)$  and so there exist distinct elements  $x$  and  $y$  in  $F(\alpha)$  which are both in  $B_1$  and  $B_2$ . Therefore there exists a block  $B_3$  distinct from  $B_1$  and  $B_2$  containing both  $x$  and  $y$ . Thus  $3 = |B_i \cap B_j| \geq |B_1 \cap B_2 \cap B_3| \geq 2$  for

any two distinct  $i, j \in \{1, 2, 3\}$ . Now by Lemma 2.4,  $B_i^\alpha = B_i$  for all  $i \in \{1, 2, 3\}$  and so  $\alpha$  is a permutation on  $B_i$ . Therefore, it follows from Lemmas 2.4 and 2.6, that  $|F(\alpha) \cap B| \in \{6, 11\}$  for all blocks  $B \in F_b(\alpha)$ . Thus  $|F(\alpha) \cap (B_1 \cup B_2 \cup B_3)| \geq 11$  and so  $|F(\alpha)| \in \{11, 16\}$ . If  $|F(\alpha)| = 16$ , then

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \setminus F(\alpha)| \geq 16 + 16 \cdot 5 = 85,$$

which is a contradiction by Lemma 2.5. If  $|F(\alpha)| = 11$ , then there is no block  $B \in F_b(\alpha)$  such that  $|F(\alpha) \cap B| = 11$ , since otherwise  $|(B' \cup B) \cap F(\alpha)| \geq 11 + 6 - 3 = 14$  for any block  $B' \in F(\alpha)$  distinct from  $B$ . Hence, in this case,

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \setminus F(\alpha)| \geq 11 + 11 \cdot 10 = 121,$$

which contradicts Lemma 2.5. This completes the proof.  $\square$

**Lemma 3.4.** *Let  $G$  be an automorphism group of a possible symmetric  $(81, 16, 3)$  design which is a 5-group. Then  $|G| \leq 5$ .*

*Proof.* It is enough to show that  $G$  has no subgroup  $H$  of order  $5^2$ . If  $\alpha \in G$  is of order 25, then by Lemma 3.3,  $|F(\alpha)| = 1$ , since  $\emptyset \neq F(\alpha) \subseteq F(\alpha^5)$ . Then, by Lemma 3.3, the number of orbits of the action of  $H$  on  $G$  is equal to

$$r = \frac{1}{5^2} \sum_{h \in H} |F(h)| = \frac{81 + 24 \cdot 1}{5^2} = \frac{21}{5}.$$

This is a contradiction, since  $r$  should be an integer.  $\square$

**Lemma 3.5.** *Let  $\alpha$  be an automorphism of a possible symmetric  $(81, 16, 3)$  design of order 13. Then  $|F(\alpha)| = 3$ .*

*Proof.* Since  $\alpha$  is a permutation on the point set, it follows from Lemma 2.2 that  $|F(\alpha)| \in \{3, 16\}$ . Suppose, for a contradiction, that  $|F(\alpha)| = |F_b(\alpha)| = 16$ . Then, by Lemma 2.6,  $|F(\alpha) \cap B| = 3$  for all  $B \in F_b(\alpha)$ . Thus

$$|F(\alpha)| + \sum_{B \in F_b(\alpha)} |B \setminus F(\alpha)| \geq 16 + 16 \cdot 13 = 224,$$

contradicting Lemma 2.5. This completes the proof.  $\square$

**Lemma 3.6.** *Let  $G$  be an automorphism group of a possible symmetric  $(81, 16, 3)$  design which is a 13-group. Then  $|G| \leq 13$ .*

*Proof.* It is enough to show that  $G$  has no subgroup  $H$  of order  $13^2$ . Since  $13^2 > 81$ ,  $G$  has no element of order  $13^2$ . Thus  $H$  is an elementary abelian 13-group. Then, by Lemma 3.5, the number of orbits of the action of  $H$  on  $G$  is equal to

$$r = \frac{1}{13^2} \sum_{h \in H} |F(h)| = \frac{81 + 12 \cdot 3}{13^2} = \frac{9}{13}.$$

This is a contradiction, since  $r$  should be an integer.  $\square$

**Lemma 3.7.** *Let  $G$  be an automorphism group of a possible symmetric  $(81, 16, 3)$  design. Then  $G$  has no element with the following orders: 10, 26, 65.*

*Proof.* (1) Suppose that  $G$  has an element of order 10. Then  $G$  contains two automorphisms  $\alpha$  and  $\beta$  of orders 5 and 2 respectively such that  $\alpha\beta = \beta\alpha$ . Since  $\alpha$  and  $\beta$  commutes,  $\alpha(F(\beta)) = F(\beta)$ . By Theorem 1.2 we have that  $|F(\beta)| = 9$ . Now by considering the cycle decomposition of  $\alpha$  on  $F(\beta)$ , it follows that  $|F(\alpha) \cap F(\beta)| \in \{4, 9\}$  which contradicts Lemma 3.3.

(2) Suppose that  $G$  has an element of order 26. Then  $G$  contains two automorphisms  $\alpha$  and  $\beta$  of orders 13 and 2 respectively such that  $\alpha\beta = \beta\alpha$ . Since  $\alpha$  and  $\beta$  commutes,  $\alpha(F(\beta)) = F(\beta)$  and by Theorem 1.2,  $|F(\beta)| = 9$ , the cycle decomposition of  $\alpha$  on  $F(\beta)$  shows that  $F(\beta) \subseteq F(\alpha)$  which contradicts Lemma 3.5.

(3) Suppose that  $G$  has an element of order 65. Then  $G$  contains two automorphisms  $\alpha$  and  $\beta$  of orders 13 and 5 respectively such that  $\alpha\beta = \beta\alpha$ . Since  $\alpha$  and  $\beta$  commutes,  $\beta(F(\alpha)) = F(\alpha)$ . But by Lemma 3.5 we have that  $|F(\alpha)| = 3$  so the cycle decomposition of  $\beta$  on  $F(\alpha)$  implies that  $F(\alpha) \subseteq F(\beta)$  which contradicts Lemma 3.3.  $\square$

**Proof of Theorem 1.4.** It follows from Lemmas 3.1, 3.2, 3.4, 3.6 and 3.7

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# Monophonic and $m$ -Hull Number of the Cartesian Product of Graphs

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## ABSTRACT

In this paper, we use the concept of block graph to characterize the monophonic sets in  $G \times K_n$ , where  $G$  is a graph containing a cutpoint and  $K_n$  is the complete graph of order  $n$ . We also give the monophonic and  $m$ -hull numbers of graphs resulting from the cartesian product of two graphs.

## 1 Introduction

Given a connected graph  $G = (V(G), E(G))$  and vertices  $u$  and  $v$  of  $G$ , we call any  $u$ - $v$  path of length  $d_G(u, v)$  (length of the shortest path connecting  $u$  and  $v$ ) as *geodesic*. Any induced path in  $G$  that does not contain an edge joining two non-consecutive vertices  $u$  and  $v$  is called a  $u$ - $v$  *monophonic path* (or simply,  $m$ -path). The *monophonic closure* of a subset  $S$  of  $V(G)$  is  $J_G[S] = \bigcup_{u,v \in S} J_G[u, v]$ , where  $J_G[u, v]$  is the set containing  $u$  and  $v$  and all vertices lying on some  $u$ - $v$   $m$ -path. If  $J_G[S] = V(G)$ , then we call  $S$  a *monophonic set* in  $G$ . A monophonic set in  $G$  of minimum order is called a *minimum monophonic set* in  $G$ . The order of a minimum monophonic set in  $G$  is called the *monophonic number* of  $G$ , denoted by  $m(G)$ .

A subset  $C$  of  $V(G)$  is said to be  $m$ -convex if, for every pair of vertices  $x, y \in C$ , the vertex set of every  $x$ - $y$   $m$ -path is contained in  $C$ . It is easy to verify that  $S$  is  $m$ -convex if and only if  $J_G[S] = S$ .

The (monophonic)  $m$ -convex hull  $[S]_m$  of a subset  $S$  of  $V(G)$  is the smallest  $m$ -convex set in  $G$  containing  $S$ . It can be formed from the sequence  $\{J_G^p[S]\}$ , where  $p$  is a nonnegative integer,  $J_G^0[S] = S$ ,  $J_G^1[S] = J[S]$ , and  $J_G^p[S] = J[J_G^{p-1}[S]]$  for  $p \geq 2$ . For some  $p$ , we must have  $J_G^q[S] = J_G^p[S]$  for all  $q \geq p$ . Further, if  $p$  is the smallest nonnegative integer such that  $J_G^q[S] = J_G^p[S]$  for