

MINIMAL NUMBER OF GENERATORS AND MINIMUM ORDER OF A NON-ABELIAN GROUP WHOSE ELEMENTS COMMUTE WITH THEIR ENDOMORPHIC IMAGES

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A group in which every element commutes with its endomorphic images is called an “E-group”. If p is a prime number, a p -group G which is an E-group is called a “ pE -group”. Every abelian group is obviously an E-group. We prove that every 2-generator E-group is abelian and that all 3-generator E-groups are nilpotent of class at most 2. It is also proved that every infinite 3-generator E-group is abelian. We conjecture that every finite 3-generator E-group should be abelian. Moreover, we show that the minimum order of a non-abelian pE -group is p^8 for any odd prime number p and this order is 2^7 for $p = 2$. Some of these results are proved for a class wider than the class of E-groups.

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1. INTRODUCTION AND RESULTS

A group in which each element commutes with its endomorphic images is called an “E-group”. It is well known (see, e.g., Malone, 1995) that a group G is an E-group if and only if the near-ring generated by the endomorphisms of G in the near-ring of maps on G is a ring.

Since in an E-group every element commutes with its image under inner automorphisms, every E-group is a 2-Engel group, and so they are nilpotent of class at most 3 (see Levi, 1942, or Robinson, 1995, Theorem 12.3.6). Throughout the article, p denotes a prime number. We call an E-group which is also a p -group, a pE -group. Since a finite E-group can be written as a direct product of its Sylow subgroups, and any direct factor of an E-group is an E-group (Malone, 1977), so we need only consider pE -groups.

The first examples of non-abelian pE -groups are due to Faudree (1971), which are defined as follows:

$$G = \langle a_1, a_2, a_3, a_4 \mid a_i^{p^2} = 1, [a_i, a_j, a_k] = 1, i, j, k \in \{1, 2, 3, 4\}, [a_1, a_2] = a_1^p, [a_1, a_3] = a_3^p, [a_1, a_4] = a_4^p, [a_2, a_3] = a_2^p, [a_2, a_4] = 1, [a_3, a_4] = a_3^p \rangle,$$

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where p is any odd prime number. Note that the above example is false for $p = 2$. This is because when $p = 2$ then the map α defined by

$$a_1^\alpha = a_1^{-1}a_2a_4, \quad a_2^\alpha = a_3, \quad a_3^\alpha = a_4, \quad a_4^\alpha = a_1a_4$$

can be extended to an endomorphism of G . But $[a_3^\alpha, a_3] = [a_4, a_3] \neq 1$, so that G is not an E -group. All known examples of non-abelian E -groups have nilpotency class 2 (see Caranti, 1985; Caranti et al., 1987; Faudree, 1971; Malone, 1980). In this article, we see new examples of E -groups.

Caranti posed the question (The Kourovka Notebook, 2002, Problem 11.46a) of whether there exists a finite $3E$ -group of nilpotency class 3. (Note that every E -group without elements of order 3 is nilpotent of class at most 2).

Some partial (negative) answers to this question are as follows: finite $3E$ -groups of exponent dividing 9 are nilpotent of class at most 2 (Malone, 1977); every 2-generator E -group is nilpotent of class at most 2 (since they are 2-Engel). Here we concentrate on the following questions:

- (1) What is the least number of generators of a finitely generated non-abelian E -group?
- (2) What is the minimum order of a finite non-abelian pE -group?

We prove the following theorem.

Theorem 1.1. *Every finite 3-generator E -group is nilpotent of class at most 2.*

Theorem 1.2. *Let G be a $3E$ -group. If $|G| \leq 3^{10}$, then G is nilpotent of class at most 2.*

As we said, it is easily seen that every 2-generator E -group is nilpotent of class at most 2. In fact a stronger result holds, namely, the following theorem.

Theorem 1.3.

- (i) *Every 2-generator group is abelian if and only if it is an E -group.*
- (ii) *Every infinite 3-generator group is abelian if and only if it is an E -group.*

Thus, by Theorem 1.3 and Faudree's examples of E -groups, the minimal number of generators of a non-abelian E -group is 3 or 4. We conjecture that this minimal number must be 4, that is every 3-generator E -group is abelian. In response to question (2), we prove the following theorem.

Theorem 1.4. *For any prime number p , every pE -group of order at most p^6 is abelian.*

Theorem 1.5.

- (i) *For any odd prime number p , every pE -group of order at most p^7 is abelian.*
- (ii) *There exist non-abelian $2E$ -groups of order 2^7 .*

As a result of Theorems 1.4 and 1.5 and Faudree’s examples of E -groups, we conclude that the minimum order of a finite non-abelian pE -group is p^8 , for any odd prime number p and this order is 2^7 for $p = 2$.

2. PRELIMINARY DEFINITIONS AND RESULTS

Let G be a group, H be an element or a subgroup of G , and n be a positive integer. We denote by $G', \Phi(G), G^n, \gamma_3(G), Z(G), Z_2(G), \exp(G), \text{Aut}(G), \text{End}(G), C_G(H), Q_8$, and C_n , respectively, the derived subgroup, the Frattini subgroup, the subgroup generated by n -powers of the elements, the third term of the lower central series, the center, the second center, the exponent, the automorphism group, the set of endomorphisms of G , the centralizer of H in G , the quaternion group of order 8, and the cyclic group of order n . For a finite p -group G and a positive integer n , we denote by $\Omega_n(G)$ the subgroup generated by elements x such that $x^{p^n} = 1$. If G is a nilpotent group, we denote by $\text{cl}(G)$ the nilpotency class of G . If G is a finite group, $d(G)$ will denote the minimum number of generators of G . If a, b , and c are elements of a group, we denote by $[a, b]$ the commutator $a^{-1}b^{-1}ab$ and we define $[a, b, c] = [[a, b], c]$, as usual. An automorphism α of a group G is called *central* if $x^{-1}x^\alpha \in Z(G)$ for all $x \in G$.

As we have found that some of our results are valid for a class of finite p -groups larger than pE -groups, we introduce the following class of p -groups for every prime number p .

Definition 2.1. A finite p -group G is called a $p\mathcal{E}$ -group if G is a 2-Engel group and there exists a non-negative integer r such that $\Omega_r(G) \leq Z(G)$ and $\exp(\frac{G}{G'}) = p^r$.

Remark 2.2. We know that a finite pE -group is a $p\mathcal{E}$ -group (Malone, 1977); but the converse is false in general as one can see that Q_8 and the group

$$G = \langle a_1, a_2, a_3 : [a_i, a_j, a_k] = 1, (i, j, k \in \{1, 2, 3\}), a_1^4 = 1, a_1^2 = a_2^2 = a_3^2 = [a_1, a_2], [a_2, a_3]^2 = [a_1, a_3]^2 = 1 \rangle$$

are $2\mathcal{E}$ -groups which are not $2E$ -groups. The quaternion group

$$Q_8 = \langle a, b \mid a^b = b^{-1}, a^2 = b^2, a^4 = 1 \rangle$$

is not an E -group, since $[a, b] \neq 1$ and the map α which sends a to b and b to a , can be extended to an automorphism of Q_8 .

Lemma 2.3. If G is a finite 2-Engel p -group, $p > 2$, and $m \in \mathbb{N}$, then $G^{p^m} = \{g^{p^m} : g \in G\}$ and G is regular.

Proof. This follows easily from the following fact which is valid for all $a, b \in G$:

$$a^{p^m} b^{p^m} = (ab)^{p^m} [a, b]^{\frac{p^m(p^m-1)}{2}} = \left(ab[a, b]^{\frac{p^m-1}{2}} \right)^{p^m}. \quad \square$$

Lemma 2.4. Let G be a $p\mathcal{E}$ -group such that $\exp(\frac{G}{G'}) = p^r$.

- (i) $\exp(G') = \exp(G/Z(G))$ and $\exp(G) = p^r \exp(G')$.

- (ii) If $\text{cl}(G) = 2$, then $\exp(G') \leq p^r$.
- (iii) If $\text{cl}(G) = 3$, then $p = 3$ and $\exp(G') = 3^{r+1}$.

Proof. (i) We have

$$\begin{aligned} \exp(G/Z(G)) \text{ divides } n &\Leftrightarrow [a^n, b] = 1 \quad \forall a, b \in G \\ &\Leftrightarrow [a, b]^n = 1 \quad \forall a, b \in G \\ &\Leftrightarrow \exp(G') \text{ divides } n. \end{aligned}$$

This shows that $\exp(G') = \exp(G/Z(G))$. For the second part of (i), note that if $\exp(G') = p^t$, then $G^{p^{r+t}} \leq (G^{p^r})^{p^t} \leq (G')^{p^t} = 1$. Also if $G^{p^{r+t-1}} = 1$, then $G^{p^{t-1}} \leq \Omega_r(G) \leq Z(G)$ and so $\exp(G') \leq p^{t-1}$, which is impossible. It follows that $\exp(G) = p^{r+t}$.

(ii) This follows from (i) and the fact that $G' \leq Z(G)$.

(iii) Note that in every 2-Engel group K , $\gamma_3(K)^3 = 1$ (see Robinson (1995, Theorem 12.3.6)). Thus, since $\text{cl}(G) = 3$, we have $p = 3$ and one can write

$$(G')^{3^{r+1}} = [G^{3^r}, G]^3 \leq [G', G]^3 = \gamma_3(G)^3 = 1$$

which gives $\exp(G') \leq 3^{r+1}$. If $\exp(G') \leq 3^r$, then $G' \leq \Omega_r(G) \leq Z(G)$ which is not possible, as $\text{cl}(G) = 3$. Therefore, $\exp(G') = 3^{r+1}$. □

Theorem 2.5. *Every 2-generator $p\mathcal{E}$ -group is either abelian or isomorphic to Q_8 .*

Proof. Suppose that $G = \langle a, b \rangle$ is a $p\mathcal{E}$ -group and that $\exp(G/G') = n = p^r$. Then $a^n, b^n \in G' = \langle [a, b] \rangle$ and since G' is a cyclic p -group, we have $\langle a^n, b^n \rangle = \langle a^n \rangle$ or $\langle b^n \rangle$. Without loss of generality, we can suppose that $b^n = a^{ns}$ for some integer s . Then

$$(ba^{-s})^n = [b, a]^{sn(n-1)/2}, \tag{1}$$

which by Lemma 2.4 is trivial if p is odd or if $p = 2$ and either $\exp(G') \leq 2^{r-1}$ or $2 \mid s$. In any of these cases we would have that ba^{-s} is in $Z(G)$ and G is abelian. So one can suppose that $p = 2$, that s is odd and that $\exp(G') = 2^r$. In this case the equality (1) implies that $(ba^{-s})^{2n} = 1$ and since G is a $2\mathcal{E}$ -group, we have $(ba^{-s})^2 \in Z(G)$. Thus

$$1 = [(ba^{-s})^2, a] = [ba^{-s}, a]^2 = [b, a]^2.$$

It follows that $r \leq 1$ and so $|G| = |\frac{G}{G'}| |G'| \leq 8$ and G is either abelian or $G \cong Q_8$ or the dihedral group D_8 of order 8. But D_8 is not a $2\mathcal{E}$ -group, since there are elements of order 2 in D_8 which are not central. On the other hand, Q_8 is a $2\mathcal{E}$ -group, since $\exp(Q_8/Q_8') = 2$ and the only element of order 2 in Q_8 is central. This completes the proof. □

Lemma 2.6. *Let G be a $p\mathcal{E}$ -group and let $\exp(\frac{G}{G'}) = p^r$. Then $Z_2(G)^{p^r} = Z(G) \cap G^{p^r}$. In particular if $\text{cl}(G) = 3$, then $\exp(\frac{Z_2(G)}{Z(G)}) = 3^r$.*

Proof. By Robinson (1995, Theorem 12.3.6), we may assume that $p = 3$. Let $x \in Z_2(G)^{3^r}$. By Lemma 2.3, $x = y^{3^r}$ for some $y \in Z_2(G)$. Since $G^{3^r} \leq G' \leq Z_2(G)$, we have

$$[x, g] = [y^{3^r}, g] = [y, g^{3^r}] = 1$$

for all $g \in G$. This implies that $x \in Z(G)$.

Now assume that $x \in G^{3^r} \cap Z(G)$. Then $x = y^{3^r}$ for some $y \in G$ and so

$$1 = [x, g] = [y^{3^r}, g] = [y, g]^{3^r}$$

for all $g \in G$. Then $[y, g] \in \Omega_r(G) \leq Z(G)$ which implies that $y \in Z_2(G)$. Hence $x \in Z_2(G)^{3^r}$. This completes the proof of the first part.

If $\text{cl}(G) = 3$ and $Z_2(G)^{3^{r-1}} \leq Z(G)$, then we have

$$G^{3^r} = (G^3)^{3^{r-1}} \leq Z_2(G)^{3^{r-1}} \leq Z(G),$$

(note that since $\gamma_3(G)^3 = 1$, $G^3 \leq Z_2(G)$). Thus $G^{3^r} \leq Z(G)$ which is impossible by Lemma 2.4(ii). Therefore $\exp(\frac{Z_2(G)}{Z(G)}) = 3^r$. □

Remark 2.7. If G is a 2-Engel group, then $G^3 G' \leq Z_2(G)$. This is because, $\text{cl}(G) \leq 3$ and $\gamma_3(G)^3 = 1$ (see Robinson, 1995, Theorem 12.3.6). Thus if G is a finite 2-Engel 3-group, then we always have that $\Phi(G) \leq Z_2(G)$.

Lemma 2.8. *Let G be a 2-Engel group. If $\frac{G}{Z_2(G)}$ is 2-generator, then $\text{cl}(G) \leq 2$.*

Proof. If $\frac{G}{Z_2(G)} = \langle aZ_2(G), bZ_2(G) \rangle$, then $G = \langle a, b, Z_2(G) \rangle$. Thus

$$G' = \langle [x, y], \gamma_3(G) \mid x, y \in \{a, b\} \cup Z_2(G) \rangle.$$

Since G is 2-Engel, we have $G' \leq Z(G)$. This completes the proof. □

Theorem 2.9. *Every 3-generator $p\mathcal{E}$ -group is nilpotent of class at most 2.*

Proof. For a contradiction suppose that $G = \langle x, y, z \rangle$ is a $p\mathcal{E}$ -group of class 3. Suppose that $\exp(G/G') = 3^r$. Let $H = (G')^3 \gamma_3(G)$. Notice that, by Lemma 2.4, $[H, G] = H^{3^r} = 1$. Modulo H we have that

$$x^{3^r} = [x, y]^\alpha [y, z]^\beta [z, x]^\gamma, \quad y^{3^r} = [x, y]^\alpha [y, z]^\beta [z, x]^\gamma, \quad z^{3^r} = [x, y]^\alpha [y, z]^\beta [z, x]^\gamma,$$

for some integers $\alpha, \beta, \gamma, \alpha', \beta', \gamma', t_1, t_2, t_3 \in \{-1, 0, 1\}$. Since $[x, y, z] \neq 1$, it follows that t_i 's must all be zero. Now since $[x^{3^r}, y] = [x, y^{3^r}]$, one can see that $\beta' = -\beta$. Similarly one can deduce that $\gamma' = -\gamma$ and $\alpha' = -\alpha$. Therefore, modulo H we have

$$x^{3^r} = [x, y]^\alpha [z, x]^\beta, \quad y^{3^r} = [x, y]^\alpha [y, z]^{-\beta} \quad z^{3^r} = [y, z]^{-\alpha} [z, x]^{-\gamma}.$$

It follows that

$$[x, y]^{3^r} = [x, y, z]^\beta, \quad [z, x]^{3^r} = [x, y, z]^{-\alpha}, \quad [y, z]^{3^r} = [x, y, z]^\gamma.$$

Then $x^{3^{2r}} = y^{3^{2r}} = z^{3^{2r}} = 1$. Now since G is regular, it follows that $G^{3^{2r}} = 1$ that contradicts Lemma 2.4. \square

Theorem 2.10. *Let G be a non-abelian 3-generator $p^{\mathcal{E}}$ -group, $\exp(\frac{G}{G'}) = p^r$, $\exp(G') = p^t$, and $p > 2$. Then $|G| = p^{3(r+t)}$ and G has the following presentation*

$$\langle x, y, z \mid x^{p^{r+t}} = y^{p^{r+t}} = z^{p^{r+t}} = [x^{p^t}, y] = [x^{p^t}, z] = [y^{p^t}, x] = [y^{p^t}, z] = [z^{p^t}, x] = [z^{p^t}, y] = 1, \\ [x, y] = x^{p^t t_{11}} y^{p^t t_{12}} z^{p^t t_{13}}, [x, z] = x^{p^t t_{21}} y^{p^t t_{22}} z^{p^t t_{23}}, [y, z] = x^{p^t t_{31}} y^{p^t t_{32}} z^{p^t t_{33}} \rangle,$$

where $1 \leq t \leq r$ and $[t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$. Moreover, every group with the above presentation is a $p^{\mathcal{E}}$ -group.

Proof. By Theorem 2.9, $cl(G) = 2$. Since G is 3-generator, there exist elements $a, b, c \in G$ such that $\frac{G}{Z(G)} = \langle aZ(G) \rangle \times \langle bZ(G) \rangle \times \langle cZ(G) \rangle$. Thus, since G' has exponent p^t , we have $|aZ(G)| = |bZ(G)| = p^t$ and $|cZ(G)| = p^s$ for some integers $t, s \geq 0$ with $t \geq s$. We also have $G' = \langle [a, b], [a, c], [b, c] \rangle$, since $cl(G) = 2$ and $G = \langle a, b, c, Z(G) \rangle$. Since $|aZ(G)| = p^t$ and $|cZ(G)| = p^s$, we have $|[a, b]| \leq p^t$, $|[a, c]| \leq p^s$ and $|[b, c]| \leq p^s$. Therefore $|G'| \leq p^{t+2s}$. Also, since G is regular, $|G : \Omega_r(G)| = |G^{p^r}|$. Then $|G| \leq |\Omega_r(G)||G^{p^r}| \leq |Z(G)||G'|$ and so $|G : Z(G)| \leq |G'|$. Hence $p^{2r+s} \leq p^{t+2s}$ and $t \leq s$. It follows that $s = t$, $|G'| = |\frac{G}{Z(G)}| = p^{3t}$ and $G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle$. Since G is not abelian, $t \geq 1$. Thus $\frac{G}{G'Z(G)} \cong C_p \times C_p \times C_p$. This implies that $G = \langle a, b, c \rangle$.

Now, since $G^{p^r} \leq G'$ and $|G' : Z(G)| \leq |G : \Omega_r(G)| = |G^{p^r}|$, we have $G' = G^{p^r}$. Also we have $G^{p^r} = \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle$ (since $t \leq r$). By Lemma 2.4 $\exp(G) = p^{r+t}$ and since $G' = G^{p^r}$ is an abelian group of order p^{3t} it follows that $G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle$, $t \leq r$ and $|a| = |b| = |c| = p^{r+t}$. Also since $G^{p^r} = \langle a^{p^r}, b^{p^r}, c^{p^r} \rangle$ and $G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \leq G^{p^r}$, it is not hard to see that $G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle$ and so

$$p^{3r} = |G^{p^r}| \leq |\Omega_r(G)| \leq |Z(G)| = |G : G'| \leq p^{3r}.$$

It follows that $G^{p^r} = \Omega_r(G) = Z(G)$, $|\Omega_r(G)| = |G : G^{p^r}| = |G'|$ and so $G' = \Omega_r(G)$. Thus we have the following information about G :

$$\begin{aligned} |G| &= p^{3(r+t)}, \exp(G) = p^{r+t}, G = \langle a, b, c \rangle, \\ |a| &= |b| = |c| = p^{r+t} \\ Z(G) &= \Omega_r(G) = G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle \\ G' &= \Omega_r(G) = G^{p^r} = \langle a^{p^r} \rangle \times \langle b^{p^r} \rangle \times \langle c^{p^r} \rangle. \end{aligned}$$

Hence there exists a 3×3 matrix $T = [t_{ij}] \in GL(3, \mathbb{Z}_{p^t})$ such that

$$[a, b] = a^{p^t t_{11}} b^{p^t t_{12}} c^{p^t t_{13}}$$

$$[a, c] = a^{p^r t_{21}} b^{p^r t_{22}} c^{p^r t_{23}}$$

$$[b, c] = a^{p^r t_{31}} b^{p^r t_{32}} c^{p^r t_{33}}$$

and every element of G can be written as $a^i b^j c^k$ for some $i, j, k \in \mathbb{Z}$ and

$$(a^i b^j c^k)(a^{i'} b^{j'} c^{k'}) = a^{i+i'-i'jp^r t_{11}-i'kp^r t_{21}-j'kp^r t_{31}}$$

$$b^{j+j'-i'jp^r t_{12}-i'kp^r t_{22}-j'kp^r t_{32}} c^{k+k'-i'jp^r t_{13}-i'kp^r t_{23}-j'kp^r t_{33}}.$$

Now consider $\tilde{G} = \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}} \times \mathbb{Z}_{p^{r+t}}$ and define the following binary operation on \tilde{G} :

$$(i, j, k)(i', j', k') = (i + i' - i'jp^r t_{11} - i'kp^r t_{21} - j'kp^r t_{31}, j + j' - i'jp^r t_{12}$$

$$- i'kp^r t_{22} - j'kp^r t_{32}, k + k' - i'jp^r t_{13} - i'kp^r t_{23} - j'kp^r t_{33})$$

It is easy to see that \tilde{G} with this binary operation is a group and $G \cong \tilde{G}$. Now one can easily see that the group G has the required presentation. □

Theorem 2.11 (The Main Result of Morigi, 1995). *For p an odd prime, there exists no finite non-abelian 3-generator p -group having an abelian automorphism group.*

Theorem 2.12. *Let G be a non-abelian finite 3-generator pE -group and $p > 2$. Then $\exp(G') < \exp(\frac{G}{G'})$.*

Proof. Suppose, for a contradiction, that $\exp(G') \geq \exp(\frac{G}{G'})$. By Lemma 2.4, $\exp(G') = \exp(\frac{G}{G'}) = p^r$ and by the proof of Theorem 2.10, we have $G = \langle a, b, c \rangle$ and

$$Z(G) = \Omega_r(G) = G^{p^r} = G' = \langle [a, b] \rangle \times \langle [a, c] \rangle \times \langle [b, c] \rangle, |a| = |b| = |c| = p^{2r},$$

$$|[a, b]| = |[a, c]| = |[b, c]| = p^r.$$

If we prove that $Aut(G)$ is abelian, then Theorem 2.11 completes the proof.

Let $\alpha \in Aut(G)$. There exist integers i, n, m , and an element $w \in G'$ such that $a^\alpha = a^i b^n c^m w$. Since $[a^\alpha, a] = 1$, we have $[b, a]^n [c, a]^m = 1$ and so $n \equiv m \equiv 0 \pmod{p^r}$. Therefore $a^\alpha = a^i w_a$ and similarly $b^\alpha = b^j w_b, c^\alpha = c^k w_c$, where $1 \leq i, j, k \leq p^r - 1$ and $w_a, w_b, w_c \in G' = Z(G)$. From $[(ab)^\alpha, ab] = 1$ and $[(ac)^\alpha, ac] = 1$, it follows, respectively, that $i = j$ and $i = k$. Also from equality $G^{p^r} = G'$, we have $a^{p^r} = [a, b]^t [b, c]^r [a, c]^s$ where t, r , and s are integers. Then $(a^\alpha)^{p^r} = [a^\alpha, b^\alpha]^t [b^\alpha, c^\alpha]^r [a^\alpha, c^\alpha]^s$ and we obtain $a^{p^r i} = x^{p^r i^2}$. Therefore, $i^2 \equiv i \pmod{p^r}$ and so $i = 1$. Therefore, all automorphisms of G are central so that they fix the elements of $G' = Z(G)$. If $\alpha, \beta \in Aut(G)$, then $x^{\alpha\beta} = x^{\beta\alpha}$ for every $x \in \{a, b, c\}$. Hence $Aut(G)$ is abelian which contradicts Theorem 2.11. □

Corollary 2.13. *Let G be a finite 3-generator pE -group and $p > 2$. If $\exp(G) \leq p^2$, then G is abelian.*

Proof. If G is non-abelian, then by Lemma 2.4, we have $\exp(G') = \exp(\frac{G}{G'}) = p$ which contradicts Theorem 2.12. \square

Lemma 2.14. *Let G be a $p\mathcal{E}$ -group such that $|G| \leq p^5$. Then G is abelian or G is isomorphic to one of the following groups:*

$$Q_8, Q_8 \times C_2, Q_8 \times C_2 \times C_2,$$

$$\langle x, y, z \mid x^4 = y^4 = [y, z] = 1, x^2 = z^2 = [x, y], (xz)^2 = y^2 \rangle, \quad \text{or}$$

$$\langle x, y, z \mid x^4 = z^4 = [y, z] = 1, x^2 = y^2 = [x, y], [x, z] = z^2 \rangle.$$

Proof. It can be easily checked by GAP (2005) that there are exactly five non-abelian $2\mathcal{E}$ -groups, which are the same as listed in the lemma. Thus we may assume that $p > 2$ and G is non-abelian. If $d(G) \geq 4$, then $|Z(G)| \geq |\Omega_1(G)| = |G : G^p| \geq p^4$ (since G is regular) and so $|G : Z(G)| \leq p$, which is impossible. Hence $d(G) = 3$ and so $|G| \geq p^4$ which contradicts Theorem 2.10. \square

Remark 2.15. Suppose that G is a finite p -group such that $\Omega_1(G) \leq Z(G)$. If G has no nontrivial abelian direct factor, then $\Omega_1(G) \leq \Phi(G)$. To see this, let $x \in G$ be of order p and $x \notin \Phi(G)$. Then there exists a maximal subgroup M such that $x \notin M$. Since $x \in Z(G)$, we have $\langle x \rangle \trianglelefteq G$, so that $G = M \times \langle x \rangle$, a contradiction.

Theorem 2.16. *Let G be a $p\mathcal{E}$ -group having no abelian direct factor and $p > 2$. If $|G| = p^7$, then G is abelian.*

Proof. Suppose, for a contradiction, that G is not abelian. If $d(G) \geq 4$, by Remark 2.15, we have

$$|\Phi(G)| \geq |\Omega_1(G)| = |G : G^p| \geq |G : \Phi(G)| \geq p^4$$

and so $|G| \geq p^8$ which is impossible. Therefore, by Lemma 2.5, we have $d(G) = 3$ which is a contradiction by Theorem 2.10. \square

Lemma 2.17. *Let G be an E -group and $a \in G$ be such that $\langle aG' \rangle$ is an infinite direct summand of $\frac{G}{G'}$. Then $a \in Z(G)$.*

Proof. By assumption, we have $\frac{G}{G'} = \langle aG' \rangle \oplus \langle XG' \rangle$ for some $X \subseteq G$. Let $\pi : G \rightarrow \frac{G}{G'}$ be the natural epimorphism and $\psi : \langle aG' \rangle \oplus \langle XG' \rangle \rightarrow \langle aG' \rangle$ the projection map on the first component. Now for each $x \in G$ let $\varphi_x : \langle aG' \rangle \rightarrow \langle x \rangle$ be the map defined by $a^i G' \mapsto x^i$ for all $i \in \mathbb{Z}$. Since $\langle aG' \rangle \cong \mathbb{Z}$, φ_x is a group homomorphism mapping aG' to x . Thus $\pi\psi\varphi_x$ is an endomorphism of G mapping a to x . Since G is an E -group, we have that $[a, x] = 1$. This completes the proof. \square

Theorem 2.18. *Let G be an infinite finitely generated E -group. Then $G = K \times H$, where K is a central torsion-free subgroup of G and H is a finite subgroup of G . In particular, if G is infinite and indecomposable, then G is infinite cyclic.*

Proof. Since G is infinite and nilpotent, $\frac{G}{G'}$ is an infinite finitely generated group. Thus

$$\frac{G}{G'} = \langle a_1 G' \rangle \oplus \cdots \oplus \langle a_n G' \rangle \oplus \langle b_1 G' \rangle \oplus \cdots \oplus \langle b_m G' \rangle \tag{*}$$

for some $a_1, \dots, a_n, b_1, \dots, b_m \in G$ such that $\langle a_i G' \rangle$ is infinite, $\langle b_i G' \rangle$ is finite and $G = \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$. By Lemma 2.17, $K = \langle a_1, \dots, a_n \rangle \leq Z(G)$. It follows that $G' = H'$, where $H = \langle b_1, \dots, b_m \rangle$. Thus $\frac{H}{H'}$ is finite and since H is a nilpotent group, H is finite. Since $K \leq Z(G)$ and we have the decomposition (*), K is a torsion-free group. It follows that $G = K \times H$. Now if G is indecomposable, we must have $H = 1$ and $n = 1$. This completes the proof. \square

Lemma 2.19. *Let G be a finite nilpotent p -group of class 3. If $|G' : G' \cap Z(G)| = p$, then $|G : Z_2(G)| = p^2$.*

Proof. Suppose $G = \langle a, b, c_1, \dots, c_r \rangle$ where $G'Z(G) = \langle [a, b] \rangle Z(G)$. By replacing c_i by a suitable $c_i a^{z_i} b^{\beta_i}$ one can assume that $[c_i, a], [c_i, b] \in Z(G)$ for $i = 1, \dots, r$. We claim that $c_1, \dots, c_r \in Z_2(G)$. For this it suffices to show that $[c_i, c_j] \in Z(G)$ for $1 \leq i < j \leq r$. Suppose

$$[c_i, c_j] = [a, b]^r z$$

with $z \in Z(G)$. As

$$1 = [a, b, c_k][b, c_k, a][c_k, a, b] = [a, b, c_k],$$

this is clear. Hence $G/Z_2(G) = \langle a, b \rangle Z_2(G)/Z_2(G)$ is of order p^2 . \square

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. Let G be a 3-generator E -group. If $\frac{G}{G'}$ is finite, since G is nilpotent, G is finite and G is a direct product of its Sylow subgroups. Every Sylow subgroup of G is endomorphic image of G and so by Malone (1977) they are at most 3-generator E -groups. In this case, Theorem 2.9 completes the proof. If $\frac{G}{G'}$ is infinite, then by the fundamental theorem of finitely generated abelian groups, we have $\frac{G}{G'} = \langle aG' \rangle \oplus \langle bG', cG' \rangle$ for some $a, b, c \in G$ such that aG' has infinite order. Thus by Lemma 2.17, $a \in Z(G)$ and since G is 2-Engel, it follows easily that $G' = \langle [b, c] \rangle$ and since G is 2-Engel, $\gamma_3(G) = \gamma_3(\langle [b, c] \rangle) = 1$. This completes the proof. \square

Proof of Theorem 1.2. Suppose, for a contradiction, that G is a finite 3E-group of the least order subject to the properties $\text{cl}(G) = 3$ and $|G| \leq 3^{10}$. Then G is indecomposable and so $\Omega_1(G) \leq \Phi(G)$, by Remark 2.15. Thus $\Omega_1(G) \leq \Phi(G) \cap Z(G)$. If $d(G) \geq 5$, then $|\frac{G}{\Phi(G)}| \geq 3^5$. Since G is regular and $\Phi(G) = G^3 G'$, $|\Phi(G) \cap Z(G)| \geq |\Omega_1(G)| = |G : G^3| \geq 3^5$ and since $\text{cl}(G) = 3$, $\Phi(G) \cap Z(G) \not\leq \Phi(G)$. It follows that $|G| \geq 3^{11}$, a contradiction. Thus Theorem 1.1 implies that $d(G) = 4$. If $|G' : Z(G) \cap G'| = 3$, then by Lemma 2.19, we have $|G : Z_2(G)| = 9$. Therefore $\frac{G}{Z_2(G)}$

is a 2-generator group and by Lemma 2.8, $cl(G) \leq 2$, a contradiction. Hence $|G' : Z(G) \cap G'| \geq 9$. Since

$$|G| = |G : \Phi(G)||\Phi(G) : \Phi(G) \cap Z(G)||\Phi(G) \cap Z(G)|$$

and

$$\begin{aligned} |\Phi(G) : \Phi(G) \cap Z(G)| &= |G'G^3 : G'G^3 \cap Z(G)| = |Z(G)G'G^3 : Z(G)| \\ &= \frac{|Z(G)G'| |G^3|}{|Z(G)G' \cap G^3| |Z(G)|}, \end{aligned}$$

we have

$$|G| = |G : \Phi(G)||G' : G' \cap Z(G)||G^3 : G'Z(G) \cap G^3||\Phi(G) \cap Z(G)| \geq 3^{10}.$$

Thus $|G| = 3^{10}$, $|\Omega_1(G)| = |\Phi(G) \cap Z(G)| = 3^4$, $|G' : Z(G) \cap G'| = 9$ and $G^3 \leq G'Z(G)$. Since $|G : G^3| = |\Omega_1(G)|$, $G^3 = \Phi(G) \leq Z_2(G)$, $\Phi(G)$ is an abelian group of order 3^6 and $d(\Phi(G)) = 4$. Hence

$$\Phi(G) \cong C_{27} \times C_3 \times C_3 \times C_3 \quad \text{or} \quad C_9 \times C_9 \times C_3 \times C_3.$$

Also we have $G'^9 = [G^3, G]^3 \leq (\gamma_3(G))^3 = 1$ and Lemma 2.4(ii) yields that $\exp(\frac{G}{G'}) = 3$. Hence by Lemma 2.6, we have $Z_2(G)^3 = \Phi(G) \cap Z(G)$. Now Lemma 2.8 implies that $d(\frac{G}{Z_2(G)}) = 3$ or 4 . Then $|Z_2(G)| = 3^6$ or 3^7 , which implies that $Z_2(G) = \Phi(G)$ or $|Z_2(G) : \Phi(G)| = 3$. If $Z_2(G) = \Phi(G)$, then $|Z_2(G)^3| = |\Phi(G) \cap Z(G)| = 9$, a contradiction. Thus $|Z_2(G) : \Phi(G)| = 3$. Since $[Z_2(G), \Phi(G)] = 1$, we have $\Phi(G) \leq Z(Z_2(G))$, $Z_2(G)$ is an abelian group and $d(Z_2(G)) = 4$. Hence

$$Z_2(G) \cong C_{81} \times C_3 \times C_3 \times C_3 \quad \text{or} \quad C_{27} \times C_9 \times C_3 \times C_3 \quad \text{or} \quad C_9 \times C_9 \times C_9 \times C_3.$$

Thus $|Z_2(G)^3| = |\Phi(G) \cap Z(G)| = 27$. This contradiction completes the proof.

Proof of Theorem 1.3. i) Let G be a 2-generator E -group. Suppose first that $\frac{G}{G'}$ is finite. Since G is nilpotent, G is finite and so it is the direct product of its Sylow subgroups. Every Sylow subgroup of G is also at most 2-generator and an E -group. Now Theorem 2.5 and Remark 2.2 imply that every Sylow subgroup of G is abelian and so G is abelian.

Therefore, we may assume that $\frac{G}{G'}$ is infinite. It follows from the fundamental theorem of finitely generated abelian groups, that $\frac{G}{G'} = \langle aG' \rangle \oplus \langle bG' \rangle$ for some $a, b \in G$ such that $\langle aG' \rangle$ is infinite. Since $G' \leq \Phi(G)$, $G = \langle a, b \rangle$ and Lemma 2.17 completes the proof of (i).

ii) Let G be an infinite 3-generator E -group. Since G is infinite and nilpotent, $\frac{G}{G'}$ is an infinite finitely generated 3-generator group. Thus $\frac{G}{G'} = \langle aG' \rangle \oplus \langle bG' \rangle \oplus \langle cG' \rangle$ for some $a, b, c \in G$ such that $\langle aG' \rangle$ is infinite and $G = \langle a, b, c \rangle$. By Lemma 2.17, $a \in Z(G)$. If either $\langle bG' \rangle$ or $\langle cG' \rangle$ is infinite, then Lemma 2.17 implies that G is abelian. Thus we may assume that $\langle bG' \rangle$ and $\langle cG' \rangle$ are both finite. It

follows that $G' = \langle [b, c] \rangle \leq H = \langle b, c \rangle$ is finite. Thus $G = \langle a \rangle \times H$, since $a \in Z(H)$ is of infinite order and H is a finite group. Hence H is a 2-generator E -group, as it is a direct factor of G . Now part (i) completes the proof. \square

Proof of Theorem 1.4. Suppose, for a contradiction, that G is a non-abelian pE -group of order p^6 . We see that non-abelian groups in Lemma 2.14 are not E -groups and so $|G| = p^6$. If $p = 2$, then one can see (e.g., by GAP, 2005) that there exist ten pE -groups T . We have checked by the package `AutPGrp` in GAP (2005), that for each such a group T , there are $\alpha \in \text{Aut}(T)$ and $x \in T$ such that $[x, x^\alpha] \neq 1$ (the automorphism α is in a set of generators given by the package for $\text{Aut}(T)$); thus they are not E -groups. Therefore, p is odd. Similarly by the proof of Theorem 2.16, we have $d(G) = 3$ (since G has no nontrivial abelian direct factor). Then by Theorem 2.10, we have $\exp(G) = p^2$. Hence by Corollary 2.13, the proof is complete.

Proof of Theorem 1.5. i) Suppose, for a contradiction, that G is a non-abelian pE -group of least order subject to the property $|G| \leq p^7$. Then by Theorem 1.4, $|G| = p^7$. By the choice of G and that every direct factor of an E -group is again an E -group, G has no abelian direct factor. Now Theorem 2.16 completes the proof.

ii) Let

$$G = \langle x, y, z, t \mid y^2 = z^2, [x, y] = [x, t] = y^2, [x, z] = z^2 t^2, \\ [y, z] = x^2, [y, t] = [z, t] = [y, z, t] = 1 \rangle.$$

We have $G' = Z(G) = G^2 = \Omega_1(G) = \{g^2 \mid g \in G\}$, $\exp(G) = 4$ and $|G| = 2^7$. By Package `AutPGrp` of GAP (2005), $\text{Aut}(G)$ is abelian and so every automorphism of G is central. Then for all $\beta \in \text{Aut}(G)$ and all $a \in G$, $[a, a^\beta] = 1$.

Now we prove that every endomorphism α of G which is not an automorphism, maps G into $Z(G)$. We denote by $\text{Ker } \alpha$ and $\text{Im } \alpha$ the kernel and image of α , respectively. We first show that if $y^2 \in \text{Ker } \alpha$, then $\text{Im } \alpha \leq Z(G)$. Since $(y^\alpha)^2 = (z^\alpha)^2 = 1$, we have y^α and $z^\alpha \in Z(G)$. Then $(x^\alpha)^2 = [y^\alpha, z^\alpha] = 1$ and so $x^\alpha \in Z(G)$. Also $(t^\alpha)^2 = (z^2 t^2)^\alpha = [x^\alpha, z^\alpha] = 1$ which implies that $t^\alpha \in Z(G)$. Therefore in this case, $\text{Im } \alpha \leq Z(G)$. Thus we can assume $y^2 \notin \text{Ker } \alpha$.

Since $\alpha \notin \text{Aut}(G)$, $\text{Ker } \alpha \cap Z(G) \neq 1$. Now it follows from the equality $G' = \Omega_1(G) = Z(G) = \{g^2 \mid g \in G\}$ that there exists an element $g \in G$ such that $1 \neq g^2 \in \text{Ker } \alpha$. Then $g^\alpha \in G'$ and $g \notin G'$. Thus $g = x^i y^j z^k t^l w$, where $0 \leq i, j, k, l \leq 1, w \in G'$ and at least one of the integers i, j, k, l is nonzero. For all $a \in G$, $[a, g]^\alpha = [a^\alpha, g^\alpha] = 1$ and so $[a, g] \in \text{Ker } \alpha$. Since $[t, g] \in \text{Ker } \alpha$, $y^{2i} \in \text{Ker } \alpha$ and so $i = 0$. Also $[y, g] \in \text{Ker } \alpha$, implies that $x^{2k} \in \text{Ker } \alpha$. If $k = 1$ then $(x^2)^\alpha = 1$ and so $x^\alpha \in Z(G)$. Therefore, $(y^2)^\alpha = [x^\alpha, t^\alpha] = 1$ which is impossible. Therefore $k = 0$. Since $[z, g] \in \text{Ker } \alpha$ and $[x, g] \in \text{Ker } \alpha$, we have $j = l = 0$ which is a contradiction and the proof is complete.

By a similar proof one can see that the following groups are non-abelian $2E$ -groups of order 2^7 .

$$\langle x, y, z, t \mid y^2 = z^2 t^2, [x, y] = [x, t] = z^2, [x, z] = t^2, [y, z] = x^2, \\ [y, t] = [z, t] = [y, z, t] = 1 \rangle.$$

$$\begin{aligned}
& [y, t] = [z, t] = [y, z, t] = 1). \\
& \langle x, y, z, t \mid y^2 = z^2, [x, z] = y^2, [x, t] = t^2 y^2, [x, y] = t^2, \\
& [y, z] = x^2 t^2, [y, t] = t^2, [z, t] = [y, z, t] = 1 \rangle.
\end{aligned}$$

These groups are not isomorphic and their automorphism groups are abelian and every endomorphism which is not an automorphism maps the group into its center. \square

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