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On groups with an irredundant 7-cover

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Abstract

A cover for a group is a collection of proper subgroups whose union is the whole group. A cover is irredundant if no proper sub-collection is also a cover and is called maximal if all its members are maximal subgroups. For an integer $n > 2$, a cover with n members is called an *n-cover*. In this paper we determine groups with a maximal irredundant 7-*cover* with core-free intersection. The intersection of an irredundant *n-cover* is known to have index bounded by a function of n, though in general the precise bound is not known. Here we prove that the exact bound is 81 when n is 7.

MSC: 20D60

1. Introduction and results

A covering or cover for a group G is a collection of subgroups of G whose union is G. We use the term *n-cover* for a cover with n members. The cover is irredundant if no proper sub-collection is also a cover, and is called maximal if all its members are maximal subgroups of G. A cover is called a core-free intersection if the core of its intersection is trivial. A cover is called a $\mathfrak{C}_n$-cover if it is a maximal irredundant n-cover with core-free intersection. We call a group G a $\mathfrak{C}_n$-group if G admits a $\mathfrak{C}_n$-cover.

Scorza [8] and Greco [6] determined the structure of all groups having an irredundant n-cover with core-free intersection for $n = 3, 4$ respectively. Bryce et al. [2] and Abdollahi et al. [1] characterized $\mathfrak{C}_n$-groups for $n = 5, 6$ respectively. Here we characterize $\mathfrak{C}_7$-groups.

Theorem A. *If G is a $\mathfrak{C}_7$-group with core-free intersection D, then G and D satisfy one of the following properties.*

(1) $G \cong (C_2)^6$ *or* $C_2 \times Sym_4$ *and* $|D| = 1$.
(2) $G \cong (C_3)^4$, $Sym_3 \times Sym_3$ *or* $(C_3)^3 \rtimes C_2$ *and* $|D| = 1$.
(3) $G \cong (C_3)^4 \rtimes C_2$ *and* $|D| = 2$.
(4) $G \cong Sym_4$ *or* $(C_2)^4 \rtimes C_3$ *and* $|D| = 1$.
(5) $G \cong (C_2)^4 \rtimes Sym_3$ *and* $|D| = 2$.

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(6) $G \cong (C_2)^6 \rtimes C_3$ *and* $|D| = 3$.
(7) $G \cong (C_2)^6 \rtimes Sym_3$ *and* $|D| = 6$.

Neumann [7] proved that if G has an irredundant n-cover then the index of the intersection of the cover in G is bounded by a function of n. Tomkinson [10] improved that bound and gave estimates for $f(n)$, the largest index $|G : D|$ over all groups G having an irredundant n-cover with intersection D. He suggested that the lower bound

$$g(n) = \begin{cases} 4 \cdot 3^{\frac{2(n-3)}{3}} & \text{if } n = 3k \\ 3^{\frac{2(n-1)}{3}} & \text{if } n = 3k+1 \\ 16 \cdot 3^{\frac{2(n-5)}{3}} & \text{if } n = 3k+2 \end{cases}$$

for $f(n)$, gives in fact the value of $f(n)$.

In [8,6,2], the value of $f(n)$ was obtained for $n = 3, 4, 5$, namely $f(3) = g(3)$, $f(4) = g(4)$ and $f(5) = g(5)$, respectively. Also Abdollahi et al. [1] have shown that $f(6) = g(6)$. Here, using the list of all $\mathfrak{C}_7$-groups and some further works we are able to prove that $f(7) = 81$. This coincides with Tomkinson's lower bound $g(7)$.

Theorem B. $f(7) = 81$.

For other aspects of covering groups by subgroups, especially for abelian groups, the reader may refer to [9], where some covering problems are closely related to combinatorial problems, including the so-called additive basis conjecture, the three-flow conjecture and a conjecture of Alon, Jaeger and Tarsi about nowhere zero vectors.

Throughout the paper for any $\mathfrak{C}_7$-group G, we always assume that $\{M_i \mid 1 \le i \le 7\}$ is a $\mathfrak{C}_7$-cover with intersection $D = \cap_{i=1}^{7} M_i$. Note that by [7], $\mathfrak{C}_7$-groups are finite and by [2, Lemma 2.2(a)] every $\mathfrak{C}_7$-group is a finite $\{2, 3, 5\}$-group.

Let us give an outline of the proofs of our main results. The proof of Theorem A is similar to that of the characterization of $\mathfrak{C}_5$ and $\mathfrak{C}_6$-groups given in [2] and [1], respectively. To characterize $\mathfrak{C}_7$-groups, we distinguish between three cases: nilpotent, semisimple and non-semisimple groups, where by a semisimple group we mean a group having no non-trivial normal abelian subgroup. The semisimple case cannot simply occur in the characterization of $\mathfrak{C}_5$-groups in [2], since these groups are soluble by [2, Lemma 2.2 (a)] and Burnside's $p^a q^b$ theorem. The techniques used in the proof of Theorem A are more or less similar to those in the characterization of $\mathfrak{C}_6$-groups [1].

The proof of Theorem B is similar to that of [2] and to the proof of Theorem D of [1]. Note that in this method, one should at least know the value of $f(7)$ on the class of $\mathfrak{C}_7$-groups as well as the value of $f(n)$ for $n \le 6$.

We use the usual notations; for example, C_n denotes the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, and the core of a subgroup H of G is denoted by H_G. Recall that a group G is a subdirect product of a family of groups $\{G_i \mid i \in I\}$ if there exists a family of normal subgroups $\{N_i \mid i \in I\}$ of G such that $\cap_{i \in I} N_i = 1$ and $G/N_i \cong G_i$ for all $i \in I$. We denote $\{1, 2, \ldots, 7\}$ by [7] and for each $m \in [7]$, $[7]^m$ will denote the set of all subsets of [7] of size m.

2. Nilpotent $\mathfrak{C}_7$-groups

The main result of this section is the characterization of all nilpotent $\mathfrak{C}_7$-groups. Before stating the main result, we quote Lemma 2.2 of [2] that will be used in the proofs repeatedly, sometimes without reference.

Lemma 2.1 (*See Lemma 2.2 of [2]*). *Let $\Gamma = \{A_i : 1 \le i \le m\}$ be an irredundant covering of a group G whose intersection of the members is D.*

(a) *If p is a prime number, x a p-element of G and* $|\{i : x \in A_i\}| = n$, *then either* $x \in D$ *or* $p \le m - n$.
(b) $\cap_{j \ne i} A_j = D$ *for all* $i \in \{1, 2, \ldots, m\}$.
(c) *If* $\cap_{i \in S} A_i = D$ *and* $|S| = n$, *then* $|\cap_{i \in T} A_i : D| \le m - n + 1$ *whenever* $|T| = n - 1$.
(d) *If Γ is maximal and U is an abelian minimal normal subgroup of G such that* $|\{i : U \subseteq A_i\}| = n$, *then either* $U \subseteq D$ *or* $|U| \le m - n$.

Theorem 2.2. *Let G be a $\mathfrak{C}_7$-group. If G is nilpotent, then* $G \cong (C_2)^6$ *or* $(C_3)^4$. *In particular, if* $(C_2)^6 = \langle a, b, c, d, e, f\rangle$, *then* $\langle b, c, d, e, f\rangle$, $\langle a, c, d, e, f\rangle$, $\langle a, b, c, d, e\rangle$, $\langle a, b, d, e, f\rangle$, $\langle a, b, c, e, f\rangle$, $\langle a, b, c, d, f\rangle$, $\langle ab, ac, ad, ef, de\rangle$ *provide a $\mathfrak{C}_7$-cover for* $(C_2)^6$ *and if* $(C_3)^4 = \langle a, b, c, d\rangle$, *then* $\langle a, c, d\rangle$, $\langle b, c, d\rangle$, $\langle a, b, c\rangle$, $\langle a, b, d\rangle$, $\langle ab^{-1}, c, d\rangle$, $\langle a, b, c^{-1}d\rangle$, $\langle ab, cd, a^{-1}bc^{-1}d\rangle$ *are members of a $\mathfrak{C}_7$-cover for* $(C_3)^6$.

Proof. We first deal with the case in which G is a p-group for some prime p. In this case the proof is similar to those of Lemma 2.1 and Proposition 2.2 of [1]. Then we argue as in the proof of Theorem A of [1] to prove that a nilpotent $\mathfrak{C}_7$-group is a p-group for some prime p. □

Note that Theorem 2.2 proves $f(7) \geq 81$. In Section 5 – after we have completed the proof of Theorem A – we will show that $f(7) \leq 81$.

3. Semisimple groups

Recall that by a semisimple group we mean a group having no non-trivial normal abelian subgroup. The main result of this section is

Theorem 3.1. *Semisimple groups do not have a $\mathfrak{C}_7$-cover.*

Remark 3.2. (1) The only primitive groups of degree 5 are C_5, $C_5 \rtimes C_2$, $C_5 \rtimes C_4$, Alt_5 and Sym_5.
(2) The only primitive groups of degree 6 are Alt_5, Alt_6, Sym_5 and Sym_6.

Proof of Theorem 3.1. Suppose, on the contrary, that G is semisimple and $\{M_1, \ldots, M_7\}$ is a $\mathfrak{C}_7$-cover with intersection D for G. Let $|G : M_i| = \alpha_i$ such that $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6 \leq \alpha_7$. Note that G is a $\{2, 3, 5\}$-group and it follows from Lemma 3.1 of [11] that $\alpha_2 \leq 6$. Also

$$\bigcap_{i\in S}(M_i)_G = 1 \quad \text{for every } S \in [7]^3, \qquad (*)$$

since the intersection $\cap_{i\in S}(M_i)_G$ contains no 5-element, and so it is a normal soluble subgroup of G, by Burnside's $p^a q^b$ theorem.

As $\alpha_2 \leq 6$, we distinguish three cases: $\alpha_2 \leq 4$, $\alpha_2 = 5$ and $\alpha_2 = 6$. In the following we discuss only the first case $\alpha_2 \leq 4$; the others are similar. The main idea is to determine the minimal normal subgroups of G, and then prove that the product of all minimal normal subgroups has order larger than $|G|$, which is impossible. For this reason, we prove for a suitable set of M_i's, each of them contains a minimal normal subgroup and the intersection of any two of such M_i's is core-free.

So suppose that $\alpha_1 \leq \alpha_2 \leq 4$. Then $(M_1)_G \cap (M_2)_G$ is non-trivial by semisimplicity of G. By applying Lemma 3.2 of [11], we have $\alpha_3 \leq 5$. If $\alpha_3 < 5$ then by $(*)$, G can be embedded in $Sym_4 \times Sym_4 \times Sym_4$, which is impossible. Thus $\alpha_3 = 5$, and so by Lemma 3.2 of [11], $\alpha_4 = \alpha_5 = \alpha_6 = \alpha_7 = 5$ and $M_i \cap M_j \subseteq M_1 \cup M_2$ for $3 \leq i < j \leq 7$. It follows that $M_i \cap M_j \subseteq M_1$ or M_2. Thus by $(*)$,

$$(M_i)_G \cap (M_j)_G = 1 \quad \text{for all distinct } i, j \in \{3, 4, 5, 6, 7\} \qquad (**)$$

and so G embeds into $Sym_5 \times Sym_5$. If $(M_i)_G = 1$, then G is a primitive group of degree 5, and so $G \cong Sym_5$ or Alt_5 by Remark 3.2. But Sym_5 and Alt_5 cannot be covered by seven proper subgroups by Lemma 7 of [3]. Therefore $(M_i)_G$ is non-trivial for every $i \geq 3$. On the other hand every minimal normal subgroup of G is isomorphic to Alt_5; for, if U is a minimal normal subgroup of G, there exists an index $i \geq 3$ such that $U \not\leq M_i$, and so $U \cap (M_i)_G = 1$. Also we have $\overline{U} := \frac{U(M_i)_G}{(M_i)_G} \lhd \frac{G}{(M_i)_G}$ and $\overline{U} \cong U$, which imply that $U \cong Alt_5$ by the semisimplicity of G and Remark 3.2. Since $(M_i)_G \neq 1$ for every $i \geq 3$, there exists a minimal normal subgroup $N_i \leq (M_i)_G$ of G. Now it follows from $(**)$ that $|N| = 60^5$, where $N = \mathrm{Dr}_{i=3}^{7} N_i$, which is a contradiction, since $60^5 = |N| \leq |G| \leq 120^2$. □

4. Proof of Theorem A

According to Theorems 2.2 and 3.1, to characterize all $\mathfrak{C}_7$-groups we need only consider non-nilpotent non-semisimple $\mathfrak{C}_7$-groups. Since G is not semisimple, G contains an abelian minimal normal subgroup U. Thus U is a normal elementary abelian subgroup of G. By Lemma 2.1 (d), $U \cong C_2$, $C_2 \times C_2$, C_3 or C_5. Hence four cases arise, according to which one of the four latter elementary abelian groups is isomorphic to U. In this way, we encounter that G may be isomorphic to a certain subdirect product of a set of primitive groups of degree ≤ 5. We collect these cases in Lemma 4.1. These cases may occur, in which we have to determine, for future purpose, i.e., the proof of Theorem B, the size of the intersection D of any $\mathfrak{C}_7$-group of G. The computational group theory package GAP [5] will be used to determine $|D|$ in Lemma 4.1; in fact we simply test whether G has any $\mathfrak{C}_7$-cover and, if so, we find all of them. There are cases for G for which we cannot use GAP [5], because of loose enough time and deficit of memory. So we have to deal with these cases by hand; they will be discussed separately in Lemmas 4.3, 4.4 and 4.7.

The results of this section together with Theorems 2.2 and 3.1 complete the proof of Theorem A.

Lemma 4.1. (1) *The following are not $\mathfrak{C}_7$-groups.*

(a) *Subdirect products of at most five symmetric groups Sym_3 of orders 108 or 18.*
(b) *Subdirect products of three C_2's and one Sym_3.*
(g) *Subdirect products of Sym_3 and Alt_4.*
(h) *Subdirect products of two C_2's and two Sym_3's with non-trivial center.*
(l) *Subdirect products of C_3, Alt_4 and C_3, Sym_4.*
(m) *Subdirect products of l cyclic groups C_3 and k symmetric groups Sym_3, where $l \leq 3$ and $k \leq 2$.*
(n) *Subdirect products of two dihedral groups of order 10.*
(p) *Subdirect products of two primitive groups of degree five of order 20.*

(2) *The following are $\mathfrak{C}_7$-groups, where D denotes the intersection of an arbitrary $\mathfrak{C}_7$-cover:*

(a) *Subdirect products of two alternating groups Alt_4 of order 48 with $|D| = 1$.*
(b) *Subdirect products of three symmetric groups Sym_4 of order* 24, 96 *and* 384 *with* $|D| = 1, 2$ *and* 6*, respectively.*
(e) *Among all subdirect products of two C_2's and two Sym_3 s, only $Sym_3 \times Sym_3$ is a $\mathfrak{C}_7$-group for which $D = 1$.*
(f) *Among all subdirect products of Sym_3 and Sym_4, there is only one $\mathfrak{C}_7$-group isomorphic to Sym_4 and $D = 1$.*
(m) *The only subdirect products of two C_2's and one Sym_4 which are $\mathfrak{C}_7$-groups are Sym_4 and $C_2 \times Sym_4$. The intersection D of an arbitrary $\mathfrak{C}_7$-cover for Sym_4 and $C_2 \times Sym_4$ is trivial.*
(n) *The only subdirect products of one C_2 and two primitive groups of degree* 4 *(which are Alt_4 and Sym_4) are Sym_4 and $C_2 \times Sym_4$, where for both of them $D = 1$.*
(p) *Among all subdirect products of three Sym_3's, $Sym_3 \times Sym_3$ is a $\mathfrak{C}_7$-group with $D = 1$ and possibly a group of the form $(C_3)^3 \rtimes C_2$ in $\mathfrak{C}_7$ with $D = 1$.*

Proof. The proof is similar to that of Lemma 4.1 of [1]. □

Lemma 4.2. *If G is a $\mathfrak{C}_7$-group and G has a minimal normal subgroup of order* 2*, then G is isomorphic to one of the following groups:* (1) $(C_2)^6$*,* (2) $C_2 \times Sym_4$*. In both cases any $\mathfrak{C}_7$-cover for G has trivial intersection.*

Proof. First note that G is a $\{2, 3, 5\}$-group. Suppose that U is a minimal normal abelian subgroup of G and $|U| = 2$. Then by Lemma 2.1, U is not contained in at least two M_i's, say $U \not\subseteq M_6, M_7$. Since U is central, $G = M_6U = M_7U$, $M_6, M_7 \trianglelefteq G$ and

$$|G : M_6| = |G : M_7| = 2. \tag{1}$$

Assume that $U \not\leq M_\ell$ for some $\ell \in \{1, 2, 3, 4, 5\}$. (#)

Without loss of generality we may suppose that $\ell = 5$. Then

$$|G : M_5| = 2. \tag{$*$}$$

Therefore G is a $\{2, 3\}$-group, and so G is soluble. By Theorem 2.2, $M_5 \cap M_6 \cap M_7$ is non-trivial, and so there exists a minimal normal subgroup V of G such that $V \leq M_5 \cap M_6 \cap M_7$. It follows that $|V| \in \{2, 3, 4\}$. We distinguish the following three cases.

Case 1. If $|V| = 2$, then V is not contained in at least two M_i's, say $V \not\subseteq M_3, M_4$. This yields that $|G : M_3| = |G : M_4| = 2$. Therefore G is a 2-group and by Theorem 2.2, $G \cong (C_2)^6$.

Case 2. If $|V| = 3$, then V is not contained in at least three M_i's, say $i = 2, 3, 4$. Thus $|G : M_i| = 3$ and $M_i \cap V = 1$ for every $i \in \{2, 3, 4\}$. If $K := M_6 \cap M_7 \cap (M_2)_G \cap (M_3)_G$ is non-trivial then there is a minimal normal subgroup W contained in K, and it follows that $|W| \leq 3$.

Subcase 1. If $|W| = 3$ then by Lemma 2.1, W is contained in none of M_1, M_4, M_5. Thus $|G : M_5| = 3$, which contradicts $(*)$.

Subcase 2. If $|W| = 2$, then $|G : M_1| = 2$. It follows that every 3-element of G belongs to $M_1 \cap M_5 \cap M_6 \cap M_7$. Suppose that x is a 3-element in $(M_i)_G$ for some $i \in \{2, 3, 4\}$. Then $x \in D$, which yields that $x \in D_G = 1$, and so $(M_i)_G$ is a 2-group. Thus $(M_i)_G$ is a Sylow 2-subgroup of $C_G(V) = V(M_i)_G$ for every $i \in \{2, 3, 4\}$. Therefore

$(M_2)_G = (M_3)_G = (M_4)_G$. Hence we have $M_5 \cap M_6 \cap M_7 \cap (M_2)_G = D_G = 1$. This implies that G is a subdirect product of three cyclic groups C_2 and one symmetric group Sym_3. But by Lemma 4.1(1)-b, such a group G cannot be a $\mathfrak{C}_7$-group.

Therefore $K = 1$ and G is a subdirect product of two cyclic groups C_2 and two symmetric groups Sym_3. Since G has a non-trivial center, by Lemma 4.1(1)-h, G is not a $\mathfrak{C}_7$-group.

Case 3. Let $|V| = 4$. Then $V \not\leq M_1, M_2, M_3, M_4$, and so $|G : M_i| = 4$. Suppose that $(M_4)_G \cap M_5 \cap M_6$ is non-trivial and T is a minimal normal subgroup contained in it. Thus $|T| \leq 4$. If $|T| = 2$, then there exist at least two indices $i, j \notin \{4, 5, 6\}$ such that T is not contained in both M_i and M_j. It follows that $|G : M_i| = |G : M_j| = 2$, which is impossible. On the other hand $|T| \neq 3$ since none of the M_i's has index 3. Therefore $|T| = 4$ and $|G : M_7| = 4$, which contradicts (1). Hence $(M_4)_G \cap M_5 \cap M_6 = 1$, and so G is a subdirect product of two cyclic groups C_2 and one symmetric group Sym_4. Now by Lemma 4.1 (2)-m, we have that $G \cong C_2 \times Sym_4$.

Thus we may suppose that the assumption (#) does not hold, and so $U \leq M_i$ for $i \in \{1, 2, 3, 4, 5\}$. We have $M_6 \cap M_7 \subseteq \cup_{i=1}^{5} M_i$. It follows that every 5-element of G (which is in $M_6 \cap M_7$) lies in at least three M_i's, and so G is a $\{2, 3\}$-group. Hence G is soluble. Therefore there is a minimal normal subgroup V of G such that $V \leq M_6 \cap M_7$. It follows that $|V| \in \{2, 3, 4\}$. Now by distinguishing three cases, $|V| = 2, 3$ or 4, the rest of the proof is as above. □

Lemma 4.3. *Let G be a group of order* 162 *that is a subdirect product of at most five symmetric groups Sym_3. Then G is a $\mathfrak{C}_7$-group with $|D| = 2$.*

Proof. We first prove that G is a $\mathfrak{C}_7$-group, by explicitly providing a $\mathfrak{C}_7$-cover for G. First note that G has a unique normal Sylow 3-subgroup P such that $P \cong (C_3)^4$. Take a $\mathfrak{C}_7$-cover (see Theorem 2.2) for P, say $P = \cup_{i=1}^{7} K_i$, where K_i is a normal subgroup of G of order 27. Now consider $L_i = \langle K_i, t\rangle$, where t is an element of order 2 in G. Then $\{L_i : 1 \leq i \leq 7\}$ is a $\mathfrak{C}_7$-cover for G.

Now consider an arbitrary $\mathfrak{C}_7$-cover $\{M_1, \ldots, M_7\}$ with intersection D for G. We shall prove $|D| = 2$. It is clear that $|G : M_i| \in \{2, 3\}$. Therefore we may assume that $|G : M_i| = 3$ for all $i \in \{2, \ldots, 7\}$. Then $P \cap M_i = (M_i)_G$, and so $P \cap D = P \cap (\cap_{i=2}^{7} M_i) = \bigcap_{i=2}^{7}(P \cap M_i) = D_G = 1$. Thus D is a 2-group. Now it is enough to show that $D \neq 1$.

Suppose, for a contradiction, that $D = 1$. Then $|\cap_{i\in T} M_i| \leq 2$ for each $T \in [7]^5$. We now distinguish the following cases:

Case 1: Suppose that P occurs in the $\mathfrak{C}_7$-cover $\{M_1, \ldots, M_7\}$. Now by considering the subcases in which (i) two members of the cover are conjugate and (ii) no two member of the cover are conjugate, one can get a contradiction in each subcase.

Case 2: By Case 1, we may assume that $|M_i| = 54$ for each $1 \leq i \leq 7$. Suppose there are two distinct i and j such that M_i and M_j are conjugate in G. As the core of the intersection of any five of the M_i's is trivial, one can prove that this case cannot happen.

Case 3: Suppose that $|M_i| = 54$ and M_i, M_j are not conjugate to each other for every two distinct $i, j \in [7]$. Now one can get a contradiction, by considering the size of $\cap_{i\in T} M_i$ ($T \in [7]^5$) and using the facts that $f(4) = 9$, $f(5) = 16$ and $f(6) = 36$. □

Lemma 4.4. *Let G be a group of order* 324 *that is a subdirect product of at most five symmetric groups Sym_3. Then G is not a $\mathfrak{C}_7$-group.*

Proof. There are three groups of order 324 up to isomorphism such that they are subdirect products of five symmetric groups Sym_3. Two of them are not $\mathfrak{C}_7$-groups. This can be checked by a similar program in the proof of Lemma 4.1 of [1]. But that program cannot be applied for the third, because of loose enough time and deficit of memory.

Suppose G has a $\mathfrak{C}_7$-cover $\{M_1, \ldots, M_7\}$ with intersection D. By hypothesis, G has a unique Sylow 3-subgroup P which is an elementary abelian group. It follows that $P \cap M_i \lhd \langle P, M_i\rangle = G$ (for all i such that $P \not\leq M_i$), and so $P \cap D = 1$. Therefore D is a 2-group. By noting that every minimal normal subgroup of G is of order 3, it is not hard to prove that $|D| = 4$. Therefore $|G : M_i| = 3$ for each $i \in [7]$ and $D = \cap_{i\in S} M_i$ for all $S \in [7]^5$. One can complete the proof in the following steps:

Step 1: M_i and M_j are not conjugate for some $i, j \in [7]$, since otherwise $|G : M_i \cap M_j| = 6$, which would imply that $|M_i \cap M_j| = 54$. On the other hand D is a subgroup of $M_i \cap M_j$ of order 4. This is a contradiction. It follows from [4, Theorem 16.2, p. 57] that $G = M_i M_j$, and so $|G : M_i \cap M_j| = 9$.

Step 2: Suppose there exists a subset $T \in [7]^4$ such that $D = \cap_{i \in T} M_i$. Since $|M_i \cap M_j \cap M_k : D| \leq 4$, $|M_i \cap M_j \cap M_k : D| = 1$ or 3. If $M_i \cap M_j \cap M_k = D$, then $|G : M_i \cap M_j \cap M_k| \leq 3^3 = 27$, a contradiction. Therefore $|M_i \cap M_j \cap M_k : D| = 3$, and so $|G : M_i \cap M_j \cap M_k| = 27$ for all distinct $i, j, k \in [7]$. It follows that $|G : M_i \cap M_j \cap M_k \cap M_t| = 27$ or 81 for all distinct $i, j, k, t \in [7]$.

Step 3: Let x and y denote the number of $S \in [7]^3$ such that $|\cap_{i \in S} M_i| = 4$ and $|\cap_{i \in S} M_i| = 12$, respectively. Then by the Inclusion–Exclusion Principle, we get $4x + 12y = 156$ and we also have $x + y = 35$. It follows that $x = 33$ and $y = 2$.

Step 4: We have $P = \cup_{i=1}^{7}(P \cap M_i)$ and $|P \cap M_i| = 27$. This implies that P has a maximal 7-cover. But it can be easily checked by GAP [5] that every 7-cover including normal subgroups of G of order 27 cannot form an irredundant cover for P. Also this cover cannot form an irredundant n-cover for $n = 5, 6$. Thus $P = \cup_{i \in T}(P \cap M_i)$ for some $T \in [7]^4$ is a maximal irredundant 4-cover. Since $f(4) = 9$, $|\cap_{i \in T}(P \cap M_i)| = 9$. But this is a contradiction, since by Step 2 we have $|\cap_{i \in T} M_i| = 4$ or 12. □

Lemma 4.5. *Let G be a $\mathfrak{C}_7$-group. If G contains a central subgroup of order* 3, *then* $G \cong (C_3)^4$.

Proof. Suppose that W is a central subgroup of order 3. Then W is not contained in at least three M_i's, say for $i = 1, 2, 3$, and so $M_i \lhd G$, yielding that $\frac{G}{M_i} \cong C_3$. Therefore every 5-element of G lies in $M_1 \cap M_2 \cap M_3$. It follows that G contains no 5-element, and so G is soluble. If $K = M_1 \cap M_2 \cap M_3$ is trivial, then $|G| \leq 27$ and G is a 3-group, which contradicts Theorem 2.2. Thus K contains a minimal normal subgroup L of G, so that $|L| \in \{2, 3, 4\}$. By Lemma 4.2, $|L| \neq 2$. If $|L| = 4$, then $L \not\leq M_i$ for $i \in \{4, 5, 6, 7\}$ and $M_1 \cap (M_4)_G = 1$. It follows that G is a subdirect product of C_3, Alt_4 or C_3, Sym_4. This contradicts Lemma 4.1(1)-l.

If $|L| = 3$, then L is not contained in at least three M_i's, say for $i = 4, 5, 6$. If L is central, then by Theorem 2.2, $G \cong (C_3)^4$; otherwise we have $\frac{G}{(M_i)_G} \cong Sym_3$ for $i = 4, 5, 6$. Now since $K \cap (M_4)_G \cap (M_5)_G = 1$, the group G is a subdirect product of at most three cyclic groups C_3 and at most two symmetric groups Sym_3, which contradicts Lemma 4.1(1)-m. This completes the proof. □

Lemma 4.6. *Let G be a $\mathfrak{C}_7$-group. Suppose that G contains a normal subgroup of order* 3. *Then G is isomorphic to one of the following groups, (here D is the intersection of any $\mathfrak{C}_7$-cover for G):*

(1) $(C_3)^4$*;* (2) $Sym_3 \times Sym_3$*;* (3) $(C_3)^3 \rtimes C_2$ *(this case may not occur);* (4) $(C_3)^4 \rtimes C_2$ *and* $|D| = 2$*. Moreover* $|G : D| \leq 81$.

Proof. First, G contains no normal subgroup of order 2 by the hypothesis and by Lemma 4.2. Suppose that U is a minimal normal subgroup of order 3. By Lemma 2.1, $U \not\leq M_i$ for $i = 1, 2, 3$, say. Then $G = UM_i$ and $|G : M_i| = 3$ for $1 \leq i \leq 3$. Every 5-element of $C_G(U) = U(M_i)_G$ lies in $\cap_{i=1}^{3}(M_i)_G$, and so lies in $D_G = 1$. Thus $C_G(U)$ is a $\{2, 3\}$-group. It follows that G is a $\{2, 3\}$-group, which implies that G is soluble. If $Z(G) \neq 1$, then G has a central subgroup L of order 3. By Lemma 4.5, $G \cong (C_3)^4$. Thus we may assume that $Z(G) = 1$. Now if $\cap_{i=1}^{3}(M_i)_G = 1$, then G is a subdirect product of three symmetric groups Sym_3. It follows from Lemma 4.1 (2)-p that $G \cong Sym_3 \times Sym_3$ or $G \cong (C_3)^3 \rtimes C_2$ with $D = 1$ in both cases. Therefore, in any case, $|G : D| \leq 81$. If $K := \cap_{i=1}^{3}(M_i)_G$ is non-trivial, then K contains a minimal normal subgroup V such that $|V| = 3$ or 4 by Lemma 4.2. If $|V| = 4$, then $|G : M_i| = 4$ for $i \in \{4, 5, 6, 7\}$ and $\frac{G}{(M_i)_G} \cong Alt_4$ or Sym_4 (note that $Z(G) = 1$). It follows that $(M_1)_G \cap (M_4)_G = 1$, and so G is a subdirect product of Sym_3 and Alt_4, or Sym_3 and Sym_4. By Lemma 4.1 (1)-g and (2)-f, $G \cong Sym_4$. This is a contradiction, since Sym_4 contains no normal subgroup of order 3.

If $|V| = 3$, then $|G : M_i| = 3$ and $C_G(V) = V(M_i)_G$ for $i = 4, 5, 6$, say. Since $\cap_{i=1}^{6}(M_i)_G = 1$, $C_G(U) \cap C_G(V)$ is a 3-group. On the other hand we have $|G : C_G(U)| = |G : C_G(V)| = 2$, and so $|G : C_G(U) \cap C_G(V)| = 2$ or 4. Since G does not contain any normal subgroup of order 2, $\cap_{i=1}^{5}(M_i)_G = 1$, and so G is a subdirect product of five symmetric groups Sym_3. Therefore G is supersoluble and $|G| = 3^t \cdot 2^k$, where $t \in \{2, 3, 4, 5\}$ and $k \in \{1, 2\}$ $(*)$. Also $D = \cap_{i=2}^{7} M_i$ is a 2-group $(**)$; for, if P is a unique normal Sylow 3-subgroup of G (note that G is supersoluble and it is well-known that a finite supersoluble group has a unique normal Sylow p-subgroup for the largest prime divisor p of the order of the group), then $P \cap M_i \lhd \langle P, M_i \rangle = G$ $(i = 1, \ldots, 6)$, and so $P \cap D \lhd G$. It follows that $P \cap D \leq D_G = 1$. We distinguish the following two cases:

Case 1: Suppose that $U \leq M_i$ for all $i \geq 4$. It follows from Lemma 3.2 of [11] that $M_2 \cap M_3 \subset \cup_{i=4}^{7} M_i$. Thus $M_2 \cap M_3 \leq M_j$ for some $j \in \{4, 5, 6, 7\}$ or $M_2 \cap M_3 \not\leq M_i$ for each $4 \leq i \leq 7$. The first case implies that

$M_1 \cap M_2 \cap M_3 \cap M_j = M_1 \cap M_2 \cap M_3$, and so $D = M_1 \cap M_2 \cap M_3 \cap M_t \cap M_s$ for some t, s distinct from j, and so $|M_1 \cap M_2 \cap M_3 : D| \leq 3$. It follows that $|G : D| \leq 81$. The second case implies that $\Gamma = \{M_2 \cap M_3 \cap M_i : 4 \leq i \leq 7\}$ is a 4-cover for $M_2 \cap M_3$. If Γ is irredundant, then $|M_2 \cap M_3 : \cap_{i=2}^{7} M_i| = |M_2 \cap M_3 : D| \leq f(4) = 9$, and so $|G : D| \leq 81$. If Γ is redundant, then $M_2 \cap M_3$ has a 3-cover, and so $|M_2 \cap M_3 : M_2 \cap M_3 \cap M_i \cap M_j \cap M_k| = f(3) = 4$ for some distinct $i, j, k \in \{4, 5, 6, 7\}$. It follows that $|M_2 \cap M_3 \cap M_i \cap M_j \cap M_k : D| \leq 2$, and so $|M_2 \cap M_3 : D| \leq 8$. Thus $|G : D| \leq 72$. In this case, we have proved that $|G : D| \leq 81$. Hence $|G| \in \{18, 36, 54, 108, 162, 324\}$ by $(*)$ and $(**)$. By 4.1(1)-a, $|G| \neq 18, 108$ and $|G| \neq 324$ by Lemma 4.4.

– If $|G| = 36$, then $G \cong Sym_3 \times Sym_3$.
– If $|G| = 54$, then $G = (C_3)^3 \rtimes C_2 = \cup_{i=1}^{7} M_i$.

Suppose that $D = \cap_{i=1}^{7} M_i$. Since G is supersoluble, $|G : M_i| = 2$ or 3. Assume that P is the normal Sylow 3-subgroup of G. Then $P \cap M_i \lhd G$, and so $P \cap D \lhd G$. It follows that $P \cap D = 1$, and so D is a 2-group. Therefore $|D| = 1$ or 2. If $|D| = 2$, then 2 divides $|M_i|$ for each $i \in [7]$, and so $|G : M_i| = 3$. Therefore $P = \cup_{i=1}^{7}(P \cap M_i) = \cup(M_i)_G$. Since $|P| = 27$, $\Gamma = \{(M_i)_G : 1 \leq i \leq 7\}$ does not form a $\mathfrak{C}_n$-cover for $n = 5, 6, 7$. Hence $P = \cup_{i \in S}(M_i)_G$ for some $S \in [7]^4$ is a maximal irredundant 4-cover. Since $f(4) = 9$, we have $|\cap_{i \in S}(M_i)_G| = 3$. This implies that $|\cap_{i \in S} M_i| = 6$. By the Inclusion–Exclusion Principle, $|\cup_{i \in S} M_i| = (4 \times 18) - (6 \times 6) + (4 \times 6) - 6 = 54$ follows, a contradiction. Thus $|D| \neq 2$, and so $D = 1$.

– If $|G| = 162$, then it follows from Lemma 4.3 that $G \cong (C_3)^4 \rtimes C_2$ and $|D| = 2$.

Case 2: Suppose that U is contained in at most three M_i's and by Case (1), we may assume that every minimal normal subgroup of G is contained in at most three M_i's. Then $\cap_{i \in T}(M_i)_G = 1$ for each $T \in [7]^4$. Since $C_G(U) = U(M_i)_G$, $C_G(U)$ is a 3-group, and so it is an elementary abelian group. Therefore $|G : C_G(U)| = 2$ since $Z(G) = 1$. Since $\cap_{i=1}^{4}(M_i)_G = 1$, G is a subdirect product of four symmetric groups Sym_3. Then $G \cong (C_3)^3 \rtimes C_2$ or $(C_3)^4 \rtimes C_2$. □

Lemma 4.7. *Let G be a group of order 192 that is a subdirect product of three alternating groups Alt_4. Then G is a $\mathfrak{C}_7$-group with $|D| = 3$, where D is the intersection of any $\mathfrak{C}_7$-cover.*

Proof. First, one may easily check that G is a $\mathfrak{C}_7$-group by using the command `ConjugacyClassesMaximalSubgroups(G);` instead of `MaximalSubgroups(G);` in the GAP program used in Lemma 4.1 of [1] (the program to test having $\mathfrak{C}_7$-covers must be modified). But we cannot obtain all $\mathfrak{C}_7$-covers for G, since the number of maximal subgroups of G is very large to run the program.

Now suppose that $\{M_1, \ldots, M_7\}$ is a $\mathfrak{C}_7$-cover with intersection D for G. Note that G has a unique normal Sylow 2-subgroup P which is elementary abelian. We claim that $M_i \neq P$ for each $i \in [7]$. Suppose, for a contradiction, that $M_1 = P \cong (C_2)^6$. Then $|M_1 \cap M_i| = 16$ for every $i \geq 2$. One can first prove that no two M_i's are conjugate in G. Thus by [4, Theorem 16.2, p. 57] $G = M_i M_j$, and so $|M_i \cap M_j| = 12$ for all distinct $i, j \in \{2, \ldots, 7\}$. Also we have $M_i \cap M_j \not\leq M_k$ for every distinct $i, j, k \in \{2, \ldots, 7\}$. If $(M_i)_G \cap (M_j)_G \leq M_k$, then $K := M_1 \cap (M_i)_G \cap (M_j)_G \cap (M_k)_G \neq 1$, and so K contains a minimal normal subgroup of order 3. A contradiction. Thus $G = (M_i \cap M_j)M_k$, which implies that $4 = |G : M_k| = |M_i \cap M_j : M_i \cap M_j \cap M_k|$. Then $|M_i \cap M_j \cap M_k| = 3$ for all distinct $i, j \in \{2, \ldots, 7\}$. Now it is easy to see that $|M_1 \cap M_i \cap M_j| = 4$ for all distinct $i, j \in \{2, \ldots, 7\}$. It follows that $|\cap_{i \in S} M_i| = 1$ for every $S \in [7]^t$, where $t \geq 5$, since 3 divides $|\cap_{i \in S} M_i|$. Now by applying the Inclusion–Exclusion Principle on $G = \cup_{i=1}^{7} M_i$, one can get a contradiction.

Therefore $M_i \neq P$ and $|G : M_i| = 4$ for all $i \in [7]$. Since $P \cap D = P \cap (\cap_{i=1}^{7} M_i) = D_G = 1$, D is a 3-group. We now claim that $D \neq 1$. Suppose, on the contrary, that $|D| = |\cap_{i=1}^{7} M_i| = |\cap_{i=2}^{7} M_i| = 1$. Then $|\cap_{i \in T} M_i| \leq 2$ for every $T \in [7]^5$. If there exists $T \in [7]^5$ such that $|\cap_{i \in T} M_i| = 2$, then $\cap_{i \in T} M_i \leq P$, and so $2 = \cap_{i \in T}(M_i \cap P) = \cap_{i \in T}(M_i)_G$, a contradiction. Therefore

$$\left|\bigcap_{i \in T} M_i\right| = 1 \quad \text{for each } T \in [7]^5. \qquad (*)$$

It follows from Lemma 2.1 that $|\cap_{i \in W} M_i| = 1$ or 3 for all sets $W \in [7]^4$. We complete the proof in the following three steps:

Step 1. If $M_2 = M_1^x$ and $M_3 = M_1^y$ for some $x, y \in G$, then $(M_1)_G = (M_2)_G = (M_3)_G$, which implies that $(M_1)_G \cap (M_4)_G = 1$, a contradiction.

Step 2. If $M_2 = M_1^g$ for some $g \in G$ and M_i, M_j are not conjugate for all distinct $i, j \geq 2$, then $|M_1 \cap M_2| = 16$ and $|M_1 \cap M_i| = |M_2 \cap M_i| = |M_i \cap M_j| = 12$ for $i, j > 2$. Now it is easy to see that $M_1 \cap M_2 = (M_1)_G = (M_2)_G$ and $|M_1 \cap M_2 \cap M_i| = |(M_1)_G \cap (M_i)_G| = 4$ for $i > 2$. Since $|M_1 \cap M_2 \cap M_3 \cap M_4| = 1$, $|M_i \cap M_j \cap M_k| \leq 4$. Thus $|M_i \cap M_j \cap M_k| = 3$ for all distinct $i, j, k \in \{1, 3, \ldots, 7\}$ or $\{2, \ldots, 7\}$. Now let x and y denote the number of sets $S \in [7]^4$ such that $|\cap_{i \in S} M_i| = 3$ and $|\cap_{i \in S} M_i| = 1$, respectively. By the Inclusion–Exclusion Principle, we have that $3x + y = 13$. We also have $x + y = 35$, which gives us a contradiction. Hence M_i and M_j are not conjugate for all distinct $i, j \geq 1$, $|M_i \cap M_j| = 12$ and $|M_i| = 48$.

Step 3. If $|\cap_{i \in X} M_i| = 3$ for all sets $X \in [7]^4$, then, by (∗), $M_i \cap M_j \not\leq M_k$ for all distinct $i, j, k \in [7]$. If $|M_i \cap M_j \cap M_k| = 6$ for some distinct $i, j, k \in [7]$, then $M_i \cap M_j \cap M_k \cap M_t = M_i \cap M_j \cap M_k \cap M_s$ for all distinct $i, j, k, s, t \in [7]$ and this is the Sylow 3-subgroup of $M_i \cap M_j \cap M_k$. Therefore $1 = D = M_i \cap M_j \cap M_k \cap M_t \cap M_s = M_i \cap M_j \cap M_k \cap M_t$, a contradiction. It follows that $|M_i \cap M_j \cap M_k| = 3$ for all distinct $i, j, k \in [7]$. This implies that $D = M_i \cap M_j \cap M_k \cap M_t \cap M_s = M_i \cap M_j \cap M_k \cap M_t = 1$, a contradiction. Thus there exists $T \in [7]^4$ such that $|\cap_{i \in T} M_i| = 1$. This implies that $|\cap_{i \in S} M_i| \leq 4$ for every $S \in [7]^3$. Thus $|M_i \cap M_j \cap M_k| = 3$ or 4 and $|M_i \cap M_j \cap M_k \cap M_t| = 1$ or 3, for if $|M_i \cap M_j \cap M_k \cap M_t| = 2$, then $|M_i \cap M_j \cap M_k| = |M_i \cap M_j \cap M_t| = 4 = |(M_i)_G \cap (M_j)_G|$. Since $|M_i \cap M_j| = 12$, we have that $M_i \cap M_j \cap M_k = M_i \cap M_j \cap M_t$ is the normal Sylow 2-subgroup of $M_i \cap M_j$, a contradiction. Also note that

$$|M_i \cap M_j \cap M_k| = 4 \Leftrightarrow |(M_i)_G \cap (M_j)_G \cap (M_k)_G| = 4, \qquad (\bullet)$$

for all $i, j, k \in [7]$. Now let x and y denote the number of $S \in [7]^3$ such that $|\cap_{i \in S} M_i| = 3$ and $|\cap_{i \in S} M_i| = 4$, respectively; and let z and w denote the number of $S \in [7]^4$ such that $|\cap_{i \in S} M_i| = 1$ and $|\cap_{i \in S} M_i| = 3$, respectively. Now the Inclusion–Exclusion Principle implies that

$$(3x + 4y) - (z + 3w) = 93 \quad \text{and} \quad x + y = z + w = 35.$$

But by (•) we have

$$64 = |P| = \left|\bigcup_{i=1}^{7}(P \cap M_i)\right| = \left|\bigcup_{i=1}(M_i)_G\right| = 7 \times 16 - (21 \times 4) + (x + 4y) - 35 + 21 - 7 + 1.$$

Therefore $x + 4y = 56$. It follows that $x = 28$ and $y = 7$, and so $z + 3w = 19$. This is a contradiction since $z + w = 35$. □

Lemma 4.8. *Let G be a $\mathfrak{C}_7$-group and suppose that G contains a minimal normal subgroup of order* 4 *and none of order* 2, *or* 3. *Then G is isomorphic to one of the following groups:* (1) Sym_4 *and* $|D| = 1$; (2) $(C_2)^4 \rtimes C_3$ *and* $|D| = 1$; (3) $(C_2)^4 \rtimes Sym_3$ *and* $|D| = 2$; (4) $(C_2)^6 \rtimes C_3$ *and* $|D| = 3$; (5) $(C_2)^6 \rtimes Sym_3$ *and* $|D| = 6$. *Moreover* $|G : D| \leq 64$, *where D is the intersection of any $\mathfrak{C}_7$-cover of G.*

Proof. First, G contains no normal subgroup of order 2 or 3 by hypothesis and Lemmas 4.2 and 4.6. Suppose that U is a minimal normal subgroup of order 4. Then U is not contained in at least four M_i's, say, $U \not\leq M_4, M_5, M_6, M_7$. So $G = UM_i$ and $|G : M_i| = 4$ for $i = 4, 5, 6, 7$. Thus $C_G(U) = U(M_i)_G$ for $4 \leq i \leq 7$, and so $C_G(U)$ is a $\{2, 3\}$-group by Lemma 2.1. Since G does not contain any normal subgroup of order 2 or 3, $\cap_{i=4}^{7}(M_i)_G = 1$, and so $C_G(U)$ is a 2-group, which implies that $C_G(U) = U(M_i)_G \cong (C_2)^n$ for some integer n.

The group G is a $\{2, 3\}$-group, since $\frac{G}{C_G(U)}$ embeds into Sym_3. Since $\Phi(G) = 1$ and $C_G(U)$ is an abelian normal subgroup of G, $G = C_G(U) \rtimes H$ such that $H \cong C_3$ or Sym_3. On the other hand we have $\frac{G}{(M_i)_G} \cong Alt_4$ or Sym_4, and so G is a subdirect product of four alternating groups Alt_4 or four symmetric groups Sym_4. Now we claim that G is a subdirect product of three alternating groups Alt_4 or a subdirect product of three symmetric groups Sym_4.

If there exists a subset $T \subset \{4, 5, 6, 7\}$ such that $|T| = 3$ and $\cap_{i \in T}(M_i)_G = 1$, then the claim holds. Assume that $\cap_{i \in T}(M_i)_G \neq 1$ for each subset $T \subset \{4, 5, 6, 7\}$ and $|T| = 3$. Then $|G : M_i| = 4$ for every $1 \leq i \leq 7$ and $(M_i)_G$ is abelian for each $i \in [7]$. Since $G = [(M_i)_G \cap (M_j)_G \cap (M_k)_G]M_t$ for every distinct $i, j, k, t \in [7]$, we have $|(M_i)_G \cap (M_j)_G \cap (M_k)_G| = |G : M_t| = 4$. Similarly we have $|(M_i)_G \cap (M_j)_G| = 16$ and $|(M_i)_G| = 64$ for every distinct $i, j \in [7]$. Thus $|C_G(U)| = 256$. On the other hand we have $C_G(U) \cap M_i = (M_i)_G$ for each

$i \in [7]$, since $C_G(U) = C_G(V)$ for every minimal normal subgroup V of G. Therefore $C_G(U) = \cup_{i=1}^{7}(M_i)_G$. By the Inclusion–Exclusion Principle, $|\cup_{i=1}^{7}(M_i)_G| = 232$. This is a contradiction. This completes the proof of the claim.

If there exists an i such that $(M_i)_G = 1$ and $|G : M_i| = 4$, then $G \cong Sym_4$. If G is a subdirect product of Alt_4 and Alt_4, then $|G| = 48$ and $|D| = 1$ by Lemma 4.1(2)-a. If G is a subdirect product of Sym_4 and Sym_4, then $|G| = 96$ and $|D| = 2$ by Lemma 4.1(2)-b.

If $G = (C_2)^6 \rtimes C_3$, then $|D| = 3$ by Lemma 4.7.

If $G = (C_2)^6 \rtimes Sym_3$, then $|D| = 6$ by Lemma 4.1(2)-b, and so $|G : D| = 64$. □

Lemma 4.9. *Let G be a $\mathfrak{C}_7$-group. Then G does not contain any normal subgroup of order* 5.

Proof. Suppose, for a contradiction, that U is a normal subgroup of G of order 5. Then U is not contained in at least five M_i's, say for $1 \le i \le 5$. Therefore $G = UM_i$, $U \cap M_i = 1$, $|G : M_i| = 5$ and $C_G(U) = U(M_i)_G$. Thus G does not contain any normal subgroup of order 3 or 4, since otherwise there would exist at least three M_i's of index 3 or four M_i's of index 4. We may assume that G contains no normal subgroup of order 2 by Lemma 4.2. $C_G(U)$ is a $\{2, 5\}$-group since every 3-element of $C_G(U)$ lies in $\cap_{i=1}^{5}(M_i)_G$, and so lies in $D_G = 1$. Hence G is a $\{2, 5\}$-group since $\frac{G}{C_G(U)}$ embeds into C_4. Since G is soluble and G contains no normal subgroup of order 2, 3 or 4, we have

$$\bigcap_{i\in S}(M_i)_G = 1 \quad \text{for every subset } S \subset \{1, \ldots, 5\} \text{ with } |S| = 3. \qquad (*)$$

Every 2-element of $C_G(U)$ lies in $\cap_{i=1}^{5}(M_i)_G = 1$. So $C_G(U)$ is the unique normal Sylow 5-subgroup of G. Thus $C_G(U)$ is an elementary abelian 5-group of rank at most 3. Also note that $\frac{G}{(M_i)_G}$ is a soluble primitive group of degree 5, and so $\frac{G}{(M_i)_G} \cong C_5$, $C_5 \rtimes C_2$ or $C_5 \rtimes C_4$. If $\frac{G}{(M_i)_G} \cong C_5$ for some $i \in \{1, \ldots, 5\}$, then U is central, and so $G = U \times M_j$ for $j \in \{1, \ldots, 5\}$. It follows that G is a 5-group by $(*)$, which is a contradiction by Theorem 2.2. Thus $\frac{G}{(M_i)_G} \cong C_5 \rtimes C_2$ or $C_5 \rtimes C_4$.

If $N := (M_1)_G \cap (M_2)_G \neq 1$, then N contains a normal subgroup of G of order 5. It follows that $|G : M_i| = 5$ for $i \in \{3, \ldots, 7\}$. Now by $(*)$, we can apply Lemma 3.2 of [11] such that $V_i := M_i$ for $i \in \{3, \ldots, 7\}$. Therefore we have $M_i \cap M_j \subseteq M_1 \cup M_2$, which implies that $(M_i)_G \cap (M_j)_G = 1$ for every distinct $i, j \in \{3, \ldots, 7\}$ by $(*)$. Hence there exist distinct $i, j \in [7]$ such that $(M_i)_G \cap (M_j)_G = 1$, and so G is a subdirect product of H and H, where $H \cong C_5 \rtimes C_2$ or $C_5 \rtimes C_4$ and $G = (C_5 \times C_5) \rtimes C_2$ or $(C_5 \times C_5) \rtimes C_4$. By Lemma 4.1(1)-n,p such groups do not have a $\mathfrak{C}_7$-cover. □

5. The value of $f(7)$

Note that we already know (from Section 2) that $f(7) \ge 81$.

Proof of Theorem B. Suppose, on the contrary, that G is a group with an irredundant 7-cover $\mathcal{C}$ with core-free intersection D such that $|G : D| > 81$.

By Theorem A, $\mathcal{C}$ is not maximal. Suppose that $\mathcal{C}$ is chosen from among such 7-covers of G with as many maximal subgroups as possible. Let $\mathcal{C}^*$ be a cover of G that we get from $\mathcal{C}$ by replacing one of its non-maximal subgroup by a maximal subgroup containing it. Let D^* be the intersection of $\mathcal{C}^*$. The cover $\mathcal{C}^*$ is redundant; for, otherwise $D^* = D$ by Lemma 2.1, and so $(D^*)_G = 1$, while $\mathcal{C}^*$ has more maximal subgroups than $\mathcal{C}$ does. It follows that we may write $G = \cup_{i=1}^{7} A_i$, where A_1 is not maximal and if A_1^* is a maximal subgroup containing it, then $\mathcal{C}^* = \{A_1^*, A_2, \ldots, A_7\}$ is redundant as a cover of G. If G is an irredundant union of six subgroups in $\mathcal{C}^*$, we may suppose that

$$G = A_1^* \cup A_2 \cup \cdots \cup A_6,$$

since A_1^* is certainly essential. If $D_1 = A_1^* \cap A_2 \cap \cdots \cap A_6$, then it follows from Theorem D of [1], that $|G : D_1| \le 36$. But $|D_1 : D| \le 2$ by Lemma 2.1. Therefore $|G : D| \le 72$, a contradiction.

If G is an irredundant union of five subgroups from $\mathcal{C}^*$, then we may suppose that

$$G = A_1^* \cup A_2 \cup \cdots \cup A_5.$$

If $D_1 = A_1^* \cap A_2 \cap A_3 \cap A_4$, then $|G : D_1| \le f(5) = 16$ by Theorem 1.1 of [2]. We know $|D_1 : D| \le 3! = 6$. Since $|G : D| > 81$ and G is a $\{2, 3\}$-group, $|G : D_1| = 16$ and $|D_1 : D| = 6$. If there exists $i \in \{2, 3, 4, 5\}$ such that

$|G : A_i| = 8$, then $D_1 = A_i \cap A_j$ for some $j \neq i$ and $2 \leq j \leq 5$. It follows that $D = \cap_{i=1}^{7} A_i = A_1 \cap A_i \cap A_j \cap A_6 \cap A_7$, and so $|D_1 : D| \leq 3$, a contradiction. If there exists $i \in \{2, 3, 4, 5\}$ such that $|G : A_i| = 4$, then $|G : A_i \cap A_j| = 8$ or 16, and so $A_i \cap A_j = A_i \cap A_j \cap A_k$ or $A_i \cap A_j = D_1$, a contradiction. Therefore $|G : A_i| = 2$ for each $i \in \{2, 3, 4, 5\}$, and so $|G : A_1^*| = 2$. In particular, $\frac{G}{D_1} \cong (C_2)^4$. It is easy to see $|A_1 \cap D_1 : D| = 2$, and so $|A_1 D_1 : A_1| = |D_1 : D_1 \cap A_1| = 3$. Thus 6 divides $|G : A_1|$. Since $|G : D| = 96$, we have $|G : A_1 \cap D_1| = 48$; this yields that $|G : A_1|$ divides 48. Therefore $|G : A_1| \in \{6, 12, 24, 48\}$. Now it is not hard to get a contradiction by considering the possible sizes obtained for the index of A_1 in G.

Now assume that G is an irredundant union of four subgroups in $\mathcal{C}^*$. We may suppose that

$$\{A_1^*, A_2, A_3, A_4\}$$

is an irredundant cover for G. If $D_1 = A_1^* \cap A_2 \cap A_3 \cap A_4 = A_2 \cap A_3 \cap A_4$, then $|G : D_1| \leq 9$ and $|G : A_1^*| \leq 3$ by [6]. Now we distinguish between the following three cases:

Case 1: Assume that $|G : D_1| = 9$ or 6. Then $D_1 = A_1^* \cap A_i$ for each $i \in \{2, 3, 4\}$ and the set $\{A_1, D_1, A_1^* \cap A_5, A_1^* \cap A_6, A_1^* \cap A_7\}$ is a cover for A_1^*. By considering the irredundancy and redundancy of the latter cover and its subcovers, one can get a contradiction.

Case 2: Assume that $|G : D_1| = 8$. Then $|G : A_1^*| = 2$ and there is an $i \in \{2, 3, 4\}$ such that A_i is not maximal. It follows that $|G : A_i| = 4$ or 8. But if $|G : A_i| = 8$, then $D_1 = A_i$, which is impossible. Therefore $|G : A_i| = 4$, and so $|G : A_1^* \cap A_i| = 4$. Thus $D_1 = A_1^* \cap A_i$. It follows that $|A_1^* : D_1| \leq 3$, and so $|G : D_1| = 4$, a contradiction.

Hence the set $\{A_1^*, A_2, A_3, A_4\}$ is a redundant cover for G, and so

$$G = A_1^* \cup A_2 \cup A_3.$$

Then $D_1 = A_1^* \cap A_2 = A_1^* \cap A_3 = A_1^* \cap A_2 \cap A_3 = A_2 \cap A_3$ and $|G : A_1^*| = 2$. Thus

$$\mathcal{D} = \{ A_1, D_1, A_1^* \cap A_4, A_1^* \cap A_5, A_1^* \cap A_6, A_1^* \cap A_7\}$$

is a cover for A_1^*. If the cover $\mathcal{D}$ is irredundant, then $|A_1^* : D| \leq 36$, and so $|G : D| \leq 72$, a contradiction. Therefore $\mathcal{D}$ is redundant, and by considering the subsets of $\mathcal{D}$ which are covers for A_1^*, one can get a contradiction. The proof is now complete. □

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