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3-REWRITABLE NILPOTENT 2-GROUPS OF CLASS 2[#]

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Let n be an integer greater than 1. A group G is said to be n-rewritable (or a Q_n-group) if for every n elements $x_1, x_2, \ldots, x_n$ in G there exist distinct permutations σ and τ in S_n such that $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)} = x_{\tau(1)}x_{\tau(2)}\cdots x_{\tau(n)}$. In this paper, we characterize all 3-rewritable nilpotent 2-groups of class 2. Also we have found a bound for the nilpotency class of certain nilpotent 3-rewritable groups, and have shown that 3-rewritable groups satisfy a certain law.

Key Words: Nilpotent groups; Permutable groups; Rewritable groups.

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1. INTRODUCTION AND RESULTS

Let n be an integer greater than 1. A group G is said to be n-rewritable (or a Q_n-group) if for every n elements $x_1, x_2, \ldots, x_n$ in G there exist distinct permutations σ and τ of the set $\{1, 2, \ldots, n\}$ such that

$$x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)} = x_{\tau(1)}x_{\tau(2)}\cdots x_{\tau(n)}.$$

We denote by Q_n, the class of all Q_n-groups. The class of 2-rewritable groups is precisely the class of abelian groups, while Q_3, Q_4, etc. are successively weaker properties.

In the above definition, if one of the permutations σ or τ can always be chosen to be the identity then the group G is said to be n-permutable (or a P_n-group) and we denote by P_n, the class of all P_n-groups. Thus $P_n \subseteq Q_n$ for all n. But for all $n > 2$, $P_n \subsetneqq Q_n$, (see Blyth, 1988a, Proposition 2.10).

We define $P = \bigcup_{n>1} P_n$ and $Q = \bigcup_{n>1} Q_n$, so again $P \subseteq Q$. A complete classification of P-groups and Q-groups are given in Curzio et al. (1988) and Blyth (1988a) respectively; namely that the classes of P-groups and Q-groups both coincide with the class of finite-by-abelian-by-finite groups. However, while there exist thorough characterizations of P-groups and Q-groups, comparatively little is known about P_n-groups and Q_n-groups for various n.

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Curzio et al. (1983) showed that a group G has the property P_3 if and only if $|G'| \leq 2$. Also Longobardi et al. (1995), Longobardi and Stonehewer (1991), Maj (1991), Maj and Stonehewer (1990) proved that a group G has the property P_4 if and only if either G' is small (no more than 8, with some additional conditions for orders 4 and 8) or G has an abelian subgroup of index 2. Blyth (1988b) has shown that Q_4-groups are soluble and Maj (1991) proved that Q_3-groups are metabelian.

Although Q_2-groups are just the abelian groups, there is no classification for Q_3-groups. Recently Blyth (2002) has shown that a finite group of odd order is in Q_3 if and only if $|G'| \leq 5$. In view of this result and Lemma 2.1 below, to study finite, nilpotent Q_3-groups we need only consider finite nilpotent 2-groups.

In this paper, among other results, we classify finite nilpotent 2-groups of class 2 with the property Q_3. In our classification of finite 2-groups G of class 2 in Q_3, we distinguish two cases: when G' has exponent 2 and respectively 4. This is similar to the classification of finite 2-groups of class 2 in P_4 presented in Longobardi and Stonehewer (1991), however Propositions 2.8 and 2.9 guarantee the existence of finite 2-groups of class 2 in $P_4 \backslash Q_3$ and in $Q_3 \backslash P_4$. Our main results are

Theorem A. *Suppose that G is a nilpotent 2-group. Suppose that G has a maximal abelian subgroup A containing G' such that G/A is not elementary abelian. If G has the property Q_3, then the nilpotency class of G is at most* 4.

Theorem B. *A nilpotent 2-group G of class 2 lies in Q_3 if and only if $|\langle x, y, z\rangle'| \leq 4$ for all $x, y, z \in G$.*

The result of Blyth (2002), in particular, implies that a finite group G of odd order with the property Q_3 satisfies the law $[G', G^2] = 1$. This of course is not true for all groups in Q_3. For example the group $G = \langle a, x \mid a^5 = x^4 = 1, a^x = a^2\rangle$ is in Q_3 but $[G', G^2] \neq 1$. For this group we have $[G', G^2] = G'$ and so $[G', G^2]^5 = 1$. In fact, we have

Theorem C. *If G is any Q_3-group, then $[G', G^2]^{480} = 1$.*

Note that A_4 the alternating group of degree 4 is in Q_3 and $[A_4', A_4^2] = A_4'$ has exponent 2. Thus the least positive integer α for which any Q_3-group G satisfies the law $[G', G^2]^\alpha = 1$ must be divisible by 10.

2. PROOFS

Lemma 2.1. *Let G be a finite nilpotent Q_3-group. Suppose that $G = H \times K$ where H is the Sylow 2-subgroup and K is the Sylow 2′-subgroup. Then either H or K is abelian.*

Proof. Suppose for a contradiction that neither H nor K is abelian. Thus there exist two elements $x, y \in K$ such that $y \notin C_G(x) = C_G(x^2)$. Now Lemma 6 of Blyth and Robinson (1995) implies that $H \times K$ does not satisfy Q_3, which is a contradiction. ☐

As we mentioned in the introduction, Lemma 2.1, together with a result of Blyth (2002), restricts our study of finite nilpotent Q_3-groups to that of 2-groups.

To prove our main results we need to prove the following.

Lemma 2.2. *Suppose that G is a Q_3-group and A is an abelian subgroup of G containing G'. Let $a, b \in A$ and $x \in G$. Then one of the following holds*

(i) $[x, b] = 1$
(ii) $[a, x] = 1$
(iii) $[b, x^2] = 1$
(iv) $[a, x]^x = [x, b]$
(v) $[x, a]^x = [x, b]$
(vi) $[a, x] = [b, x]$
(vii) $[a, x] = [x, b]$
(viii) $[x, b] = [b, x]^x[x, a]^x$
(ix) $[x, b] = [b, x]^x[a, x]^x$.

Proof. Considering the 3-set $\{ax, x, b\}$, by 3-rewritability we have fifteen equalities between the 3-products of the set. Now using the fact that the elements a and b are in the normal abelian subgroup A, it is straightforward to see that we are left with the nine cases stated in the lemma. ☐

Lemma 2.3. *Let G be a Q_3-group and let A be an abelian subgroup of G containing G'. Suppose also that $G = A\langle x\rangle$. If $[b, x^2] \neq 1$ for some $b \in A$, then $G' = \langle [b, x]\rangle^G$.*

Proof. Let $b \in A$ be such that $[b, x^2] \neq 1$, so that $[b, x] \neq 1$. Let $a \in A$ and use Lemma 2.2 and the fact that $G' = [A, x]$ to establish the lemma. ☐

Lemma 2.4. *Let G be a finite 2-group in Q_3 and let A be an abelian subgroup of G containing G'. If $G = A\langle x\rangle$, for some $x \in G$, then one of the following holds*:

(1) $[A, x^2] = 1$;
(2) $G' \cong C_2$;
(3) $G' \cong V_4$;
(4) $G' \cong C_4$ *and* $G' \leq Z(G)$.

Proof. Suppose that $G \in Q_3$ and $x^2 \notin C_G(A)$. If there is an element $g \in (Z_2(G) \cap A)\backslash Z(G)$, such that $[g, x^2] \neq 1$, then by Lemma 2.3, $G' = \langle [g, x]\rangle^G$ and since $[g, x] \in Z(G)$, we have $G' = \langle [g, x]\rangle$ and so G is nilpotent of class at most 2. Thus, by the proof of Proposition 5.1 of Blyth (1988a), $[g, x]^4 = 1$, so in this case, (2) or (4) holds. Now assume that $[h, x^2] = 1$ for all $h \in (Z_2(G) \cap A)\backslash Z(G)$. Choose an element b in $(Z_2(G) \cap A)\backslash Z(G)$ of minimal order. Then

$$1 = [b^2, x] = [b, x^2] \qquad \text{and} \qquad [b, x] \neq 1.$$

Now let a be an element of A such that $[a, x^2] \neq 1$. We claim that

(i) $[a, x^2] = [b, x]$ and
(ii) if $c \in A$ and $[c, x^2] = 1$ with $[c, x] \neq 1$, then $[c, x] = [a, x^2]$.

For, taking c for a and a for b in Lemma 2.2, we have $[c, x]^x = [x, a]$, $[x, c]^x = [x, a]$, $[c, x] = [a, x]$, $[c, x] = [x, a]$, $[x^2, a] = [x, c]^x$ or $[x^2, a] = [c, x]^x$. Now since $[c, x^2] = 1$ and $[a, x^2] \neq 1$, the only possibilities are the last two cases, that is, $[x^2, a] = [x, c]^x$ or $[x^2, a] = [c, x]^x$. Now for a moment, suppose that $c = b$. Thus by the above, $[x^2, a] \in Z(G)$, so $[c, x] \in Z(G)$ for every c which satisfies the properties cited in (ii). Thus $[c, x]^2 = 1$ and it follows that both possibilities are equivalent to $[a, x^2] = [c, x]$, as required.

From (i) we have $A = \langle a \rangle C_A(x^2)$. We distinguish two possibilities:

Case (a) Suppose that $[a^2, x] \neq 1$. Then, by (i), $[a^2, x^2] = 1$; and, by (ii), $[a^2, x] = [a, x^2]$. Therefore $[a, x] \in Z(G)$ and $|[a, x]| = 4$. Again by (ii), $[C_A(x^2), x] \leq \langle [a^2, x] \rangle$ and so $G' = [A, x] = \langle [a, x] \rangle$, i.e. (4) holds.

Case (b) Suppose that $[a^2, x] = 1$. Then

$$[b, x] = [a, x^2] = [a, x, x]$$

and $V_4 \cong \langle [a, x], [b, x] \rangle \lhd G$. By (ii), $[C_A(x^2), x] \leq \langle [a, x^2] \rangle = \langle [b, x] \rangle$ and it follows that $G' = \langle [a, x], [b, x] \rangle$, i.e. (3) holds. □

Now we are ready to prove Theorem A.

Proof of Theorem A. Since G/A is not elementary abelian, there exists an element x in G such that $x^2 \notin A$. Let $H = A\langle x \rangle$. Then $H \lhd G$ and $|H'| \leq 4$, by Lemma 2.4. Thus $H' \leq Z_2(G)$ and so $[A, x] \leq Z_2(G)$. Now let $y \in G$. If $y^2 \notin A$, then similarly $[A, y] \leq Z_2(G)$. If $y^2 \in A$, then $(xy)^2 \notin A$ and so $[A, xy] \leq Z_2(G)$, that is $[A, y] \leq Z_2(G)$. Hence $G' \leq A \leq Z_3(G)$ and so G has nilpotency class ≤ 4. □

Following Philip Hall we call a group diabelian if it is the product of two abelian subgroups.

Lemma 2.5. *Let G be a diabelian, Q_3 nilpotent 2-group of class 2 with exp$(G') = 4$. Then $G' \cong C_4$.*

Proof. We have $G = AX$ with A and X abelian. Since the product of any abelian subgroup with the centre is abelian again, we may assume that A and X both contain $Z(G)$ and so $Z(G) \leq A \cap X$. Let $a \in A$, $x \in X$ such that $|[a, x]| = 4$. Then $[a, x^2] \neq 1$ and so, by Lemma 2.3, $[A, x] \leq \langle [a, x] \rangle$. Similarly $[a, X] \leq \langle [a, x] \rangle$. Now for each $x_1 \in X$, either $|[a, x_1]| = 4$ or $|[a, xx_1]| = 4$. Thus either $[A, x_1] \leq \langle [a, x_1] \rangle$ or $[A, xx_1] \leq \langle [a, xx_1] \rangle$. Therefore $G' = [A, X] = \langle [a, x] \rangle$. □

The following is similar to 3.1.2 of Longobardi and Stonehewer (1991) about P_4-groups, and proves a part of Theorem B.

Proposition 2.6. *Let G be a nilpotent 2-group of class 2 with exp$(G') = 4$. Then $G \in Q_3$ if and only if $G' \cong C_4$.*

Proof. If $G' \cong C_4$ then Proposition 2.4 of Blyth (1988a) yields that $G \in Q_3$. Now assume that $G \in Q_3$. Let $a, x \in G$, such that $|[a, x]| = 4$. Then

$$\langle a, x, y\rangle' = \langle [a, x]\rangle \qquad \text{for all } y \in G. \tag{1}$$

For, write $X = \langle a, x, y\rangle$ and $b = [x, y]$. Suppose that $|b| \leq 2$. By Lemma 2.5, it suffices to shows that

$$X \text{ is diabelian.} \tag{2}$$

Thus we may assume that $b \neq 1$. If $[a, x^2] \in \langle b\rangle$, then $[x, a^2y] = 1$ and (2) follows. Assume therefore that $[a, x^2] \notin \langle b\rangle$. Since $X/\langle b\rangle$ is diabelian, Lemma 2.5 gives $(X/\langle b\rangle)' = \langle [a, x]\langle b\rangle\rangle$ and so

$$[a, y] \in \langle [a, x], b\rangle \cong C_4 \times C_2.$$

If $[a, y] = [a, x]^i$ for some integer i, then $[a, y^{-1}x^i] = 1$ and

$$X = \langle a, y^{-1}x^i\rangle\langle x\rangle Z(X)$$

is diabelian. If $[a, y] = [a, x]^i b$, then $[xa^{-1}, a^{-i}y] = 1$ and again

$$X = \langle xa^{-1}, a^{-i}y\rangle\langle x\rangle Z(X)$$

is diabelian. Now suppose that $|b| = 4$. Then $|[x, y^2]| = 2$ and by the previous case

$$[x, y^2] \in \langle a, x, y^2\rangle' = \langle [a, x]\rangle.$$

Therefore $[a, y^2] = [a^2, x]$ and $[x, ay]^2 = 1$. Thus again by the previous case (with y replaced by ay)

$$X' = \langle a, x, ay\rangle' = \langle [a, x]\rangle.$$

Now we have established (1).

Let $g, z \in G$. It suffices to show that $[g, z] \in \langle [a, x]\rangle$. By (1),

$$[a, g] \quad \text{and} \quad [x, z] \;\text{ belong to } \langle [a, x]\rangle.$$

If $|[a, g]| = 4$, then again by (1)

$$\langle a, g, z\rangle' = \langle [a, g]\rangle = \langle [a, x]\rangle$$

and so $[g, z] \in \langle [a, x]\rangle$. If $|[a, g]| \leq 2$, then $|[a, xg]| = 4$ and (1) gives

$$\langle a, xg, z\rangle' = \langle [a, xg]\rangle = \langle [a, x]\rangle$$

and hence $[g, z] = [x, z]^{-1}[xg, z] \in \langle [a, x]\rangle$. □

The following proves another part of Theorem B.

Proposition 2.7. *Let G be a nilpotent 2-group of class 2 with $\exp(G') = 2$. Then $G \in Q_3$ if and only if $|\langle x, y, z\rangle'| \leq 4$, for all $x, y, z \in G$.*

Proof. If $|\langle x, y, z\rangle'| \leq 4$, for all $x, y, z \in G$, then Proposition 2.4 of Blyth (1988a) yields that $G \in Q_3$. Now assume $G \in Q_3$ and suppose, for a contradiction, that there exist $x, y, z \in G$ such that $|\langle x, y, z\rangle'| \geq 8$. Since $\langle x, y, z\rangle' = \langle [x, y], [y, z], [x, z]\rangle$ and $\exp(G') = 2$, $|\langle x, y, z\rangle'| = 8$. But the product of x, y and z should be rewritable, so two of the following products must be equal:

$$\begin{aligned}
xyz &= xyz \\
xzy &= xyz[y, z] \\
yxz &= xyz[x, y] \\
yzx &= xyz[x, y][x, z] \\
zxy &= xyz[x, z][y, z] \\
zyx &= xyz[x, y][x, z][y, z]
\end{aligned}$$

which is impossible, since $[x, y], [y, z], [x, z]$ are independent. □

Proof of Theorem B. By the proof of Proposition 5.1 of Blyth (1988a), $\exp(G') = 4$ or $\exp(G') = 2$. Now the proof follows from Propositions 2.6 and 2.7. □

Proposition 2.8. *Let G be a finite 2-group of class 2. If $\exp(G') = 2$, $|G'| = 8$ and $G/Z(G)$ can be generated by 3 elements, then $G \in P_4 \backslash Q_3$. Such group exists.*

Proof. By Theorem B of Longobardi and Stonehewer (1991), $G \in P_4$. By hypothesis $G = \langle a, b, c\rangle Z(G)$ for some $a, b, c \in G$. Thus $H' = G'$ where $H = \langle a, b, c\rangle$. Now if G were in Q_3, Proposition 2.7 would imply that $|G'| \leq 4$, which is a contradiction. Thus G belongs to $P_4 \backslash Q_3$.
Consider the group K with the generators x, y, z and the following relations:

$$\begin{aligned}
&[a, b, c] = 1 \qquad \text{for all } a, b, c \in \{x, y, z\} \\
&x^2 = y^2 = z^2 = 1
\end{aligned}$$

This is easy to see (e.g. by GAP4, 2002) that K is a 2-group of class 2 whose derived subgroup is an elementary abelian group of order 8 and so by the first part $K \in P_4 \backslash Q_3$. □

Proposition 2.9. *Any finite 2-group of class 2 with the property P_4 whose derived subgroup is of exponent 4 has the property Q_3. There are finite 2-groups in $Q_3 \backslash P_4$ which are of class 2 and whose derived subgroups have exponent 4.*

Proof. The first part follows from Proposition 2.6 and Theorem A of Longobardi and Stonehewer (1991).

Consider the group G with the generators x, y, z, t and the following relations

$$[a, b, c] = 1 \qquad \text{for all } a, b, c \in \{x, y, z, t\}$$
$$x^4 = y^8 = z^4 = t^4 = 1$$
$$[x, y] = [x, z] = [x, t] = [z, y] = [t, y] = [t, z]$$

It can be easily checked (for example by GAP4, 2002) that G has the following properties:

G is a 2-group of class 2 with $G' \cong C_4$ and all the maximal subgroups of G have derived subgroups of order 4. Now Proposition 2.6 and Theorem A of Longobardi and Stonehewer (1991) show that $G \in Q_3 \backslash P_4$. □

Proof of Theorem C. Let x be any element of G'. Since G' is abelian (see Maj, 1991), it is enough to show that $[x, y^2]^{480} = 1$ for all $y \in G$. We may consider G' as a $\mathbb{Z}G$-module and write it additively. So it is equivalent to prove that $480x(y^2 - 1) = 0$ for all $y \in G$ or equivalently we show that $480(y^2 - 1) = 0$ in the endomorphism ring of G'. Since $G \in Q_3$, for all pair of integers $(i, i_1), (j, j_1)$ and (k, k_1), the elements $x^i y^{i_1}$, $x^j y^{j_1}$ and $x^k y^{k_1}$ should be rewritable. We can write the product $(x^i y^{i_1})(x^j y^{j_1})(x^k y^{k_1})$ as follows:

$$x^{i+j+k} y^{i_1+j_1+k_1} f((i, i_1), (j, j_1), (k, k_1))$$

where f is defined additively by

$$\begin{aligned} &f((i, i_1), (j, j_1), (k, k_1)) \\ &\quad := -(k(y^{i_1} - 1) + k(y^{i_1} - 1)(y^{k_1} - 1) + k(y^{i_1} - 1)(y^{j_1} - 1)(y^{k_1} - 1) \\ &\qquad + k(y^{i_1} - 1)(y^{j_1} - 1) + j(y^{i_1} - 1) + j(y^{i_1} - 1)(y^{k_1} - 1) \\ &\qquad + j(y^{i_1} - 1)(y^{j_1} - 1)(y^{k_1} - 1) + j(y^{i_1} - 1)(y^{j_1} - 1) \\ &\qquad + k(y^{j_1} - 1) + k(y^{j_1} - 1)(y^{k_1} - 1)). \end{aligned}$$

The fifteen possible rewritable cases give us fifteen polynomials, say

$$g_{\sigma\tau}(i, i_1, j, j_1, k, k_1) = f(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) - f(X_{\tau(1)}, X_{\tau(2)}, X_{\tau(3)}),$$

one of which should be zero where $X_1 = (i, i_1)$, $X_2 = (j, j_1)$ and $X_3 = (k, k_1)$ and σ and τ are two distinct permutations in S_3. Therefore the least common multiple of these polynomials must be zero. We denote by $g(i, i_1, j, j_1, k, k_1)$ the least common multiple of all the polynomials $g_{\sigma\tau}$ where $\{\sigma, \tau\}$ runs over all the 2-subsets of S_3. Now by computing $g(4, 1, 5, 1, 3, 1)$ and $g(2, 1, 1, 1, 4, 1)$ (e.g. by MAPLE) and using the fact that every element of G is an invertible element of the group ring $\mathbb{Z}G$, (so if $ya = 0$ where $y \in G$ and $a \in G'$, then $a = 0$), we have, for all $y \in G$,

$$2(2y + 1)(y + 1)(y - 1)^2(y + 2) = 0,$$
$$6(3y + 2)(3y + 1)(2y - 1)(y - 2)(2y + 3)(y + 1)(y + 3)(y - 1) = 0.$$

Let $h_1(y) = 6(3y+2)(3y+1)(2y-1)(y-2)(2y+3)(y+1)(y+3)(y-1)$ and $h_2(y) = 2(2y+1)(y+1)(y-1)^2(y+2)$. We have

$$h_1(y) = h_2(y)(54y^3 + 81y^2 - 231y - 129) + h_3(y),$$

where $h_3(y) = 300 - 1050y^3 + 750y^2 + 1050y - 1050y^4$ and $525h_2(y) = -(2y+1)h_3(y) + h_4(y)$ where

$$h_4(y) = 2400 - 4800y^3 - 2400y^2 + 4800y.$$

Also $64h_3(y) = h_4(y)(14y+7) + h_5(y)$ where $h_5(y) = 2400 - 2400y^2$. Hence $h_5(y) = 0$ and so

$$2400(y^2 - 1) = 0 \qquad \text{for all } y \in G \quad (I).$$

Now by computing $g(5,1,2,1,7,1)$ and $g(6,1,7,1,3,1)$, we get that two polynomials $k_1(y) = 30(2y-3)(2y+5)(3y+5)(5y+3)(y+1)(5y+2)(3y-2)(y-1)$ and $k_2(y) = 12(3y-1)(3y+4)(y+4)(y-3)(y+1)(4y+3)(4y+1)(y-1)$ are zero. We can write $8k_1(y) = 125k_2(y) + k_3(y)$ where

$$k_3(y) = 1649340y^6 + 3298680y^5 - 3298680y^3 - 1649340y^2,$$

and

$$137445k_2(y) = k_3(y)(144y^2 + 144y - 1765) + k_4(y)$$

where

$$k_4(y) = -237504960 + 712514880y^3 - 475009920y^2 - 712514880y + 712514880y^4.$$

Again we may write $1296k_3(y) = k_4(y)(3y^2 + 3y - 1) + k_5(y)$ where $k_5(y) = 237504960y^2 - 237504960$. Therefore $k_5(y) = 0$ and so

$$237504960(y^2 - 1) = 0 \qquad \text{for all } y \in G \quad (II).$$

Since $\gcd(2400, 237504960) = 480$, the identities (I) and (II) imply that $480(y^2 - 1) = 0$ for all $y \in G$ as required. □

3. ACKNOWLEDGMENT

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