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Left 3-Engel elements in groups☆

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Abstract

In this paper we study left 3-Engel elements in groups. In particular, we prove that for any prime p and any left 3-Engel element x of finite p-power order in a group G, x^p is in the Baer radical of G. Also it is proved that $\langle x, y\rangle$ is nilpotent of class at most 4 for every two left 3-Engel elements in a group G.

MSC: 20F45

1. Introduction and results

Let G be any group and n be a non-negative integer. For any two elements a and b of G we define inductively $[a,_n b]$ the n-Engel commutator of the pair (a,b), as follows:

$$[a,_0 b] := a, \quad [a,b] := a^{-1}b^{-1}ab \quad \text{and} \quad [a,_n b] := [[a,_{n-1} b], b] \quad \text{for all } n>0.$$

An element x of G is called a left n-Engel element if $[g,_n x] = 1$ for all $g \in G$. We denote by $L_n(G)$, the set of all left n-Engel elements of G. So $L_0(G)=1$, $L_1(G)=Z(G)$ the centre of G, and it can be easily seen that

$$L_2(G) = \{x \in G \mid \langle x\rangle^G \text{ is abelian}\},$$

where $\langle x\rangle^G$ denotes the normal closure of x in G. Therefore $L_2(G)$ is contained in $B(G)$ the Baer radical of G, and in particular it is contained in $HP(G)$ the Hirsch–Plotkin radical of G. In general for an arbitrary group K it is not necessary that $L_n(K) \subseteq HP(K)$. Suppose, for a contradiction, that $L_n(K) \subseteq HP(K)$ for all n and all

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groups K. By a deep result of Ivanov [1], there is a finitely generated infinite group M of exponent 2^k for some positive integer k. Suppose that k is the least integer with this property, so every finitely generated group of exponent dividing 2^{k-1} is finite. Let a be any element of M of order 2 and x an arbitrary element of M. It is easy to see that $[x,_m a]=[x,a]^{(-2)^{m-1}}$ for all positive integers m. Thus every element of order 2 of M is in $L_{k+1}(M)$. So by hypothesis, $M/HP(M)$ is of exponent dividing 2^{k-1} and so it is finite. Since M is finitely generated, $HP(G)$ is also. But this yields that $HP(G)$ is a periodic finitely generated nilpotent group and so it is finite. It follows that M is finite, a contradiction.

Hence the question which naturally arises, is that: what is the least positive integer n such that $L_n(G) \nsubseteq HP(G)$? To study this question one should first study the case $n=3$, since

$$L_0(G) \subseteq L_1(G) \subseteq L_2(G) \subseteq \cdots \subseteq L_n(G) \subseteq \cdots.$$

The main object of this paper is to study $L_3(G)$. As far as we know there is no example of a group G for which $L_3(G) \nsubseteq HP(G)$. The corresponding subset to $L_n(G)$ which can be similarly defined, is $R_n(G)$ the set of all right n-Engel elements of G: an element x of G is called a right n-Engel element if $[x,_n g]=1$ for all $g \in G$. There is a well-known relation between these two subsets of G due to Heineken [4, Theorem 7.11]: $(R_n(G))^{-1} \subseteq L_{n+1}(G)$ for all integers $n>0$. It is clear that $R_0(G)=1$, $R_1(G)=Z(G)$ and by a result of Kappe [4, Corollary 1, Theorem 7.13], $R_2(G)$ is a characteristic subgroup of G. It is known also that $R_2(G) \subseteq L_2(G)$ [4, Theorem 7.13(i)]. Recently Newell [3] has shown that the normal closure of every element of $R_3(G)$ is nilpotent of class at most 3. This shows, of course, that $R_3(G) \subseteq B(G) \leqslant HP(G)$. An early example was given by Macdonald [2] shows that the inverse and square of a right 3-Engel element need not be right 3-Engel, thus $R_3(G)$ is not necessarily a subgroup. But as we show in Corollary 2.2, $L_3(G)$ is closed under taking all powers. It is interesting to note that $(R_3(G))^2 \subseteq L_3(G)$ and $(R_3(G))^4 \subseteq R_3(G)$ [3, Remark to Theorem 1].

The main results of this paper are the followings.

Theorem 1.1. *Let G be a group and p be a prime number. If $x \in L_3(G)$ and $x^{p^n}=1$ for some integer $n>1$, then $\langle x^p\rangle^G$ is soluble of derived length at most $n-1$ and x^p belongs to $B(G)$, the Baer radical of G. In particular, x^p belongs to $HP(G)$, the Hirsch–Plotkin radical of G.*

Theorem 1.2. *Let G be any group and $a,b \in L_3(G)$. Then $\langle a,b\rangle$ is nilpotent of class at most 4.*

2. Proofs

Lemma 2.1. *Suppose that G is an arbitrary group and $x,y \in G$. Then $[y,_3 x]=[y^{-1},_3 x]=1$ if and only if $\langle x,x^y\rangle$ is nilpotent of class at most 2.*

Proof. Since

$$[y^{\varepsilon},_3 x] = [x^{-1},[x^{-1},[x^{-1},y^{\varepsilon}]]]^{x^3},$$

where $\varepsilon \in \{-1,1\}$, we have

$$[y,_3 x] = [y^{-1},_3 x] = 1$$

$$\Leftrightarrow \; [x^{-1},[x^{-1},[x^{-1},y]]] = [x^{-1},[x^{-1},[x^{-1},y^{-1}]]] = 1$$

$$\Leftrightarrow \; [x^{-x^{-y}},x^{-1}] = [x^{-x^{-y^{-1}}},x^{-1}] = 1$$

$$\Leftrightarrow \; [x^{-y},_2 x^{-1}] = [x^{-y^{-1}},_2 x^{-1}] = 1$$

$$\Leftrightarrow \; [x^{-y},_2 x^{-1}] = [x^{-1},_2 x^{-y}] = 1$$

$$\Leftrightarrow \; \langle x^{-1},x^{-y}\rangle = \langle x,x^{y}\rangle \text{ is nilpotent of class at most 2.} \quad \Box$$

So we have the following characterization of left 3-Engel elements in a group. We denote by $\mathcal{N}_2$ the class of nilpotent groups of class at most 2.

Corollary 2.2. *For an arbitrary group* G,

$$L_3(G) = \{x \in G \mid \langle x,x^y\rangle \in \mathcal{N}_2 \text{ for all } y \in G\}.$$

In particular every power of a left 3-*Engel element is also a left* 3-*Engel element.*

Proposition 2.3. *A group generated by a set of left* 3-*Engel elements of finite orders such that their orders are pairwise coprime is abelian.*

Proof. It is enough to show that $[a,b]=1$ for any two left 3-Engel elements a and b such that $\gcd(|a|,|b|)=1$. By Corollary 2.2, $K=\langle a,a^b\rangle$ and $H=\langle b,b^a\rangle$ are both nilpotent. Thus $[a,b]=a^{-1}a^b=(b^{-1})^a b \in K\cap H$. Since $\gcd(|a|,|b|)=1$ and H and K are both nilpotent, $\gcd(|H|,|K|)=1$. It follows that $[a,b]=1$, as required. $\Box$

Lemma 2.4. *Let* p *be a prime number,* G *be a group and* $x \in L_3(G)$. *If* $x^{p^n}=1$ *for some integer* $n \geqslant 2$, *then* $x^{p^{n-1}} \in L_2(G)$.

Proof. Let y be an arbitrary element of G. By Corollary 2.2, $\langle x,x^y\rangle$ is nilpotent of class at most 2. Thus

$$[(x^{-y})^{p^{n-1}},x^{p^{n-1}}]$$

$$[(x^{-y})^{p^{n-2}},x^{p^{n-1}}]^p = \text{ since } [x^{-y},x] \in Z(\langle x,x^y\rangle)$$

$$[(x^{-y})^{p^{n-2}},x^{p^n}] = 1.$$

But $[y,x^{p^{n-1}},x^{p^{n-1}}] = [(x^{-y})^{p^{n-1}},x^{p^{n-1}}] = 1$. This completes the proof. $\Box$

Proof of Theorem 1.1. By Lemma 2.4 and Corollary 2.2

$$1 \trianglelefteq \langle x^{p^{n-1}}\rangle^G \trianglelefteq \langle x^{p^{n-2}}\rangle^G \trianglelefteq \cdots \trianglelefteq \langle x^p\rangle^G$$

is a series of normal subgroups of G with abelian factors. This implies that $K = \langle x^p\rangle^G$ is soluble of derived length at most $n-1$. By Corollary 2.2, x^p and so all its conjugates in G belong to $L_3(G)$ and in particular they are in $L_3(K)$. Now a result of Gruenberg [4, Theorem 7.35] implies that $B(K)=K$. But $K \trianglelefteq G$ which yields that $x^p \in K \leqslant B(G) \leqslant HP(G)$. □

In the following calculations, one must be careful with notation. As usual $u^{g_1+g_2}$ is shorthand notation for $u^{g_1}u^{g_2}$. This means that

$$u^{(g_1+g_2)(h_1+h_2)} = u^{(g_1+g_2)h_1}u^{(g_1+g_2)h_2},$$

which does not have to be equal to $u^{g_1(h_1+h_2)}u^{g_2(h_1+h_2)}$. We also have that

$$u^{(g_1+g_2)(-h)} = ((u^{g_1}u^{g_2})^{-1})^h,$$

which is equal to $u^{-g_2h-g_1h}$. This does not have to be the same as $u^{-g_1h-g_2h}$.

The following remark easily follows from Corollary 2.2. We use in sequel this remark, sometimes without any reference.

Remark 1. Let G be any group and $a \in L_3(G)$. Then

$$[a,x]^{a^2} = [a,x]^{2a-1} \quad \text{and} \quad [a,x]^{a^{-1}} = [a,x]^{-a+2} \quad \text{for all } x \in G.$$

Lemma 2.5. *Let G be any group and $a,b \in L_3(G)$. Then*

$$[\langle a,b\rangle,\langle a,b\rangle] = \langle [a,b],[a,b]^a,[a,b]^b,[a,b]^{ab}\rangle.$$

Proof. It is enough to show that $N=\langle [a,b],[a,b]^a,[a,b]^b,[a,b]^{ab}\rangle$ is a normal subgroup of $\langle a,b\rangle$. We have

$$[a,b]^{a^2} = [a,b]^{2a-1} \in N,$$

$$[a,b]^{ba} = [a,b]^{ab[b,a]} = [a,b]^{1+ab-1} \in N,$$

$$[a,b]^{aba} = [a,b]^{a^2b[b,a]} = [a,b]^{1+a^2b-1} = [a,b]^{1+(2a-1)b-1} = [a,b]^{1+2ab-b-1} \in N.$$

Thus $N^a \leqslant N$ (I). Also we can write

$$[a,b]^{a^{-1}} = [a,b]^{-a+2} \in N,$$

$$[a,b]^{ba^{-1}} = [a,b]^{a^{-1}b[b,a^{-1}]} = [a,b]^{-a+a^{-1}b-a^{-1}} = [a,b]^{-a-ab+2b+a^{-1}} \in N,$$

$$[a,b]^{aba^{-1}} = [a,b]^{b[b,a^{-1}]} = [a,b]^{-a^{-1}+b-a^{-1}} \in N.$$

Therefore $N^{a^{-1}} \leqslant N$ (II). It follows from (I) and (II) that $N=N^a$. Now since $[a,b]^{ab} = [a,b]^{-1+ba+1} \in N$ we have $M = \langle [b,a],[b,a]^b,[b,a]^a,[b,a]^{ba}\rangle$ is equal to N. Thus by the symmetry between a and b, we have $M^b = M$ and so $N^b = N$. Hence $N \trianglelefteq \langle a,b\rangle$. This completes the proof. □

Lemma 2.6. *Let G be any group and $a,b \in L_3(G)$. Then both of the commutators $[a,b]$ and $[a,b]^{ab}$ belong to $C_G([a,b]^a,[a,b]^b)$.*

Proof. By Lemma 2.1, $\langle a, a^b \rangle \in \mathcal{N}_2$ and $\langle b, b^a \rangle \in \mathcal{N}_2$, so $[[a,b],[a,b]^a] = 1$ (I) and $[[a,b],[a,b]^b] = 1$ (II). It follows from (I) that $[[a,b]^b,[a,b]^{ab}] = 1$ and (II) yields that $[[a,b]^a,[a,b]^{ba}] = 1$ (III). But $[a,b]^{ba} = [a,b]^{1+ab-1}$ so (I) and (III) imply that $[[a,b]^a,[a,b]^{ab}] = 1$. This completes the proof. □

Proof of Theorem 1.2. We first prove that the derived subgroup of $\langle a,b \rangle$ is abelian. Since $\langle a,b \rangle'$ is the normal closure of $[a,b]$ in $\langle a,b \rangle$, it is enough to show that $[a,b]$ is in the centre of $\langle a,b \rangle'$ and by Lemma 2.6, it suffices to prove that $[[a,b],[a,b]^{ab}] = 1$.

Let $a^b = a_1$, $a^{b^2} = a_2$ and let $c = [a,b]$. Then $\langle a, a_1 \rangle$, $\langle a_1, a_2 \rangle$ and $\langle a_1, a_2 \rangle$ are each nilpotent of class at most 2, by Lemma 2.1.

Now $c = a^{-1}a_1$ and $c^b = a_1^{-1}a_2$ and these commute by Lemma 2.1. But

$$1 = [c^b, c] = [a_1^{-1}a_2, a^{-1}a_1] = [a_1^{-1}a_2, a_1][a_1^{-1}a_2, a^{-1}]^{a_1},$$

$$1 = [a_2, a_1][a_1, a]^{a_2 a_1}[a_2, a]^{-a_1},$$

$$1 = [a_2, a_1][a_1, a][a, a_2].$$

Hence $[a_1, a, a_2] = 1$. Now $[a_1, a] = [a, b, a] = [c, a]$ and $a_2 = ac^2[c, b]$. Since $[c, a]$ commutes with a and c, it follows that $[[c,a],[c,b]] = 1$. Now It follows from Lemma 2.6 that $[c^a, c^b] = 1$. Now by Remark 1 we have $c^{b^{-1}} = c^{-b}c^2$. It follows that $[c^{ab}, c] = [c^a, c^{b^{-1}}] = 1$.

Thus $\langle a,b \rangle$ is metabelian. Now for completing the proof, it is enough to show that $[a,b,x_1,x_2,x_3] = 1$ $(*)$ for all $x_1,x_2,x_3 \in \{a,b\}$. Since $\langle a,b \rangle$ is metabelian we have $[a,b,x_1,x_2,x_3] = [a,b,x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all permutations σ on the set $\{1,2,3\}$. Now $a,b \in L_3(G)$, it is easy to see that the equality $(*)$ is satisfied by all $x_1,x_2,x_3 \in \{a,b\}$. This completes the proof. □

The author could not prove a similar result to Theorem 1.2 for groups generated by 3 left 3-Engel elements. We finish this paper by the following question.

Question. Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that every nilpotent group generated by d left 3-Engel elements is nilpotent of class at most $f(d)$?

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