

SOME ENGEL CONDITIONS ON
INFINITE SUBSETS OF CERTAIN GROUPS

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Let k be a positive integer. We denote by $\mathcal{E}_k(\infty)$ the class of all groups in which every infinite subset contains two distinct elements x, y such that $[x, {}_k y] = 1$. We say that a group G is an \mathcal{E}_k^* -group provided that whenever X, Y are infinite subsets of G , there exists $x \in X, y \in Y$ such that $[x, {}_k y] = 1$. Here we prove that:

- (1) If G is a finitely generated soluble group, then $G \in \mathcal{E}_3(\infty)$ if and only if G is finite by a nilpotent group in which every two generator subgroup is nilpotent of class at most 3.
- (2) If G is a finitely generated metabelian group, then $G \in \mathcal{E}_k(\infty)$ if and only if $G/Z_k(G)$ is finite, where $Z_k(G)$ is the $(k+1)$ -th term of the upper central series of G .
- (3) If G is a finitely generated soluble $\mathcal{E}_k(\infty)$ -group, then there exists a positive integer t depending only on k such that $G/Z_t(G)$ is finite.
- (4) If G is an infinite \mathcal{E}_k^* -group in which every non-trivial finitely generated subgroup has a non-trivial finite quotient, then G is k -Engel. In particular, G is locally nilpotent.

1. INTRODUCTION AND RESULTS

Paul Erdős posed the following question [16]: Let G be an infinite group. If there is no infinite subset of G whose elements do not mutually commute, is there then a finite bound on the cardinality of each such set of elements?

The affirmative answer to this question was obtained by B.H. Neumann who proved in [16] that a group is centre-by-finite if and only if every infinite subset of the group contains two different commuting elements.

Further questions of a similar nature, with slightly different aspects, have been studied by many people (see [1, 2, 3, 4, 5, 6, 7, 13, 14]).

For a group G we denote by $Z_n(G)$ and $\gamma_n(G)$, respectively, the $(n+1)$ -th term of the upper central series and the n -th term of the lower central series of G . For $x, y, x_1, \dots, x_n \in G$ we write

$$[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2, \quad [x_1, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n],$$

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$$[x, {}_0 y] = x. \quad [x, {}_n y] = [[x, {}_{n-1} y], y].$$

Recall that a group G is said to be n -Engel if $[x, {}_n y] = 1$ for all x, y in G . For k a positive integer, let \mathcal{N}_k be the class of nilpotent groups of class at most k ; let \mathcal{F} be the class of finite groups and \mathcal{N} be the class of all nilpotent groups. We denote by (\mathcal{N}, ∞) ((\mathcal{N}_k, ∞)) the class of all groups in which every infinite subset contains two distinct elements x, y such that $\langle x, y \rangle$ is nilpotent (nilpotent of class at most k , respectively). We also denote by $\mathcal{E}_k(\infty)$ ($\mathcal{E}(\infty)$) the class of all groups in which every infinite subset contains two distinct elements x, y such that $[x, {}_k y] = 1$ ($[x, {}_n y] = 1$ for some positive integer n depending on x, y , respectively) and denote by $\mathcal{N}_k^{(2)}$ the class of all groups in which every 2-generator subgroup is nilpotent of class at most k .

Lennox and Wiegold [13] proved that a finitely generated soluble group G is in (\mathcal{N}, ∞) if and only if G is \mathcal{FN} . In [3] and [4] Delizia proved that a finitely generated soluble group or finitely generated residually finite group G is in (\mathcal{N}_2, ∞) if and only if $G/Z_2(G)$ is finite. Longobardi and Maj [14] proved that a finitely generated soluble group G belongs to $\mathcal{E}(\infty)$ if and only if G is \mathcal{FN} . Also it is proved in [14] that a finitely generated soluble group G belongs to $\mathcal{E}_2(\infty)$ if and only if $G/R(G)$ is finite, where $R(G)$ denotes the characteristic subgroup of G consisting of all right 2-Engel elements of G . Abdollahi [1] improved the later result by proving that a finitely generated soluble group G belongs to $\mathcal{E}_2(\infty)$ if and only if $G/Z_2(G)$ is finite. In fact this result shows that on the class of finitely generated soluble groups, we have $\mathcal{E}_2(\infty) = (\mathcal{N}_2, \infty)$. Here we prove that on the class of finitely generated soluble groups, we also have $\mathcal{E}_3(\infty) = (\mathcal{N}_3, \infty)$, by proving

THEOREM 1. *Let G be a finitely generated soluble group. Then $G \in \mathcal{E}_3(\infty)$ if and only if G is $\mathcal{FN}_3^{(2)}$.*

Abdollahi and Taeri [2] studied the class (\mathcal{N}_k, ∞) and proved that a finitely generated soluble group G is in (\mathcal{N}_k, ∞) if and only if G is $\mathcal{FN}_k^{(2)}$. Also, they proved that a finitely generated metabelian group G is in (\mathcal{N}_k, ∞) if and only if $G/Z_k(G)$ is finite. Here we extend the later result to the class of $\mathcal{E}_k(\infty)$ (Theorem 2, below). In [2] it is remarked that if $G/Z_k(G)$ is finite then G is $\mathcal{FN}_k^{(2)}$ but the converse is false for $k \geq 3$, even if G is finitely generated and soluble of derived length three. The examples cited, which are due to Newman [17], are torsion-free nilpotent.

THEOREM 2. *Let G be a finitely generated metabelian group. Then $G \in \mathcal{E}_k(\infty)$ if and only if $G/Z_k(G)$ is finite.*

By [2, Lemma 2], if G is a torsion-free nilpotent (\mathcal{N}_k, ∞) -group then G belongs to $\mathcal{N}_k^{(2)}$, and so G is k -Engel. By a result of Zel'manov [20], G is nilpotent of class at most $f(k)$, where $f(k)$ is a function of k and independent of the number of generators of G . We prove a similar result about the torsion-free nilpotent groups in the class $\mathcal{E}_k(\infty)$ (Lemma 4, below), from which we obtain

THEOREM 3. *Let G be a finitely generated soluble group which belongs to $\mathcal{E}_k(\infty)$. Then there exists a positive integer t depending only on k such that $G/Z_t(G)$ is finite.*

Let us recall that a group G is said to be locally graded whenever every finitely generated non-trivial subgroup of G has a non-trivial finite quotient. Delizia, Rhemtulla and Smith [5] recently showed that if G is a finitely generated locally graded group and $G \in (\mathcal{N}_k, \infty)$ then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite. We have been unable to prove a result similar to that of [5] about finitely generated locally graded $\mathcal{E}_k(\infty)$ -groups, but we obtain a result as follows.

Let k be a positive integer. We say that a group G is an \mathcal{E}_k^* -group provided that whenever X, Y are infinite subsets of G , there exists $x \in X, y \in Y$ such that $[x, y] = 1$. In [18], Puglisi and Spiezia proved that every infinite locally finite or locally soluble \mathcal{E}_k^* -group is a k -Engel group. We improve this result as follows.

THEOREM 4. *Let G be an infinite locally graded \mathcal{E}_k^* -group. Then G is k -Engel. In particular, G is locally nilpotent.*

2. PROOFS

We need the following easy lemma in the proofs of both Theorems 1 and 2.

LEMMA 1. *Let G be a group. Suppose that $y, x_1, \dots, x_k \in Z_k(G)$ and $a, b, c, d \in Z_4(G)$. Then for all $i \in \{1, 2, \dots, k\}$ and for all integers n .*

- (1) $[x_1, \dots, x_{i-1}, x_i y, x_{i+1}, \dots, x_k] = [x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k] \times [x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_k],$
- (2) $[x_1, \dots, x_{i-1}, x_i^n, x_{i+1}, \dots, x_k] = [x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k]^n,$
- (3) $[a, b, c, d] = [b, a, c, d]^{-1}.$

Also

- (i) *If G is metabelian then for all $x_1, \dots, x_k \in G$*

$$[x_1, x_2, \dots, x_k] = [x_2, x_1, x_3, \dots, x_k]^{-1}.$$

- (ii) *For all permutation σ on the set $\{1, \dots, k\}$, for all $a \in \gamma_2(G)$ and x_1, \dots, x_k in G , $[a, x_1, x_2, \dots, x_k] = [a, x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}].$*

PROOF: One can check the proofs of parts (1)-(3) of the lemma by some formulas of the commutator calculus. For the proofs of parts (i) and (ii) we note that $\gamma_2(G)$ is Abelian and $[a, b, c] = [a, c, b]$ for all $a \in \gamma_2(G)$ and $b, c \in G$. Thus we use the latter equality to permute the symbols in positions 3 to k of the commutator in the left hand side of (ii). \square

We use the following lemma for the proof of Theorem 1. In the proof of this lemma, we use a result of Gupta and Newman (see [10, Theorem 3.5]) which asserts that every n -generator 2-torsion-free third Engel group is nilpotent of class at most $2n - 1$.

LEMMA 2. *Every torsion-free nilpotent $\mathcal{E}_3(\infty)$ -group belongs to $\mathcal{N}_3^{(2)}$.*

PROOF: Let H be a two-generator torsion-free nilpotent $\mathcal{E}_3(\infty)$ -group. By induction on the nilpotency class of H , we may assume $H = Z_4(H)$. We prove that H is 3-Engel and so by [10, Theorem 3.5], H is nilpotent of class at most 3. Let a, b be non-trivial elements of H ; we show that $[b, a, a, a] = 1$. Consider the infinite subset $\{ab, a^2b, a^3b, \dots\}$. Since $G \in \mathcal{E}_3(\infty)$, there exist distinct positive integers i, j such that $[a^ib, a^jb, a^jb, a^jb] = 1$. Therefore by Lemma 1 (parts (1)-(3))

$$\begin{aligned} 1 &= [a^ib, a^jb, a^jb, a^jb] = [a^i, b, a^j, a^jb][a^i, b, b, a^jb][b, a^j, a^j, a^jb][b, a^j, b, a^jb] \\ &= [a, b, a, a]^{ij^2} [a, b, a, b]^{ij} [a, b, b, a]^{ij} [a, b, b, b]^i \\ &\quad \times [b, a, a, a]^{j^3} [b, a, a, b]^{j^2} [b, a, b, b]^j [b, a, b, a]^{j^2} \\ &= [b, a, a, a]^{-ij^2} [b, a, a, b]^{-ij} [a, b, b, a]^{ij} [a, b, b, b]^i \\ &\quad \times [b, a, a, a]^{j^3} [b, a, a, b]^{j^2} [a, b, b, b]^{-j} [a, b, b, a]^{-j^2} \\ &= [b, a, a, a^{j^3-ij^2} b^{j^2-ij}] [a, b, b, a^{ij-j^2} b^{i-j}] \\ &= ([b, a, a, a^{j^2} b^j] [a, b, b, a^jb]^{-1})^{j-i}. \end{aligned}$$

Since H is torsion-free, $[b, a, a, a^{j^2} b^j] [a, b, b, a^jb]^{-1} = 1$, and so $[a, b, b, a]^j [b, a, a, a]^{j^2} [b, a, a, b]^j = [a, b, b, b]^{-1}$. By arguing as above on the infinite set $\{a^{j+1}b, a^{j+2}b, \dots\}$ we get

$$[a, b, b, a]^t [b, a, a, a]^{t^2} [b, a, a, b]^t = [a, b, b, b]^{-1},$$

for some positive integer $t > j$. Therefore, by the last two equalities, we have $[b, a, a, a]^{t+j} = [a, b, b, a]^{-1} [b, a, a, b]^{-1}$.

By considering the infinite set $\{a^{t+1}b, a^{t+2}b, \dots\}$ and arguing as before, we obtain an integer $s > t$ such that $[b, a, a, a]^{s+j} = [a, b, b, a]^{-1} [b, a, a, b]^{-1}$.

Hence $[b, a, a, a]^{s+j} = [b, a, a, a]^{t+j}$ and so $[b, a, a, a] = 1$, this completes the proof. \square

PROOF OF THEOREM 1: By the result of [2], it suffices to prove that every finitely generated soluble $\mathcal{E}_3(\infty)$ -group G is $\mathcal{FN}_3^{(2)}$. By [14, Theorem 1], there exists a finite normal subgroup H of G such that G/H is nilpotent. Let T/H be the torsion subgroup of G/H . Then T is finite and G/T is a finitely generated torsion-free nilpotent group. Thus by Lemma 2, $G/T \in \mathcal{N}_3^{(2)}$ and the proof is complete. \square

COROLLARY 1. *Let G be an n -generator soluble $\mathcal{E}_3(\infty)$ -group. Then $G/Z_{2n-1}(G)$ is finite. In particular, every two-generator soluble group G belongs to $\mathcal{E}_3(\infty)$ if and only if $G/Z_3(G)$ is finite.*

PROOF: By Theorem 1, G has a finite normal subgroup T such that G/T is a torsion-free n -generator $\mathcal{N}_3^{(2)}$ -group. Thus by [10, Theorem 3.5], G/T is nilpotent of class at most $2n - 1$. Therefore $\gamma_{2n}(G)$ is finite, and hence $G/Z_{2n-1}(G)$ is finite [11]. \square

We need the following key lemma in the proof of Theorem 2. In the proof of this lemma, we use a result of Gruenberg (see [8, Theorem 1.10] or Gupta and Newman [9]),

which implies that every torsion-free metabelian k -Engel group is nilpotent of class at most k .

LEMMA 3. *Every torsion-free nilpotent metabelian group in $\mathcal{E}_k(\infty)$ is nilpotent of class at most k .*

PROOF: Let G be a torsion-free nilpotent metabelian $\mathcal{E}_k(\infty)$ -group. By induction on the nilpotency class of G , we may assume $G = Z_{k+1}(G)$. We prove that G is a k -Engel group and then by a result of Gruenberg (see [8, Theorem 1.10] or Gupta and Newman [9]), G is nilpotent of class at most k . Let x and y be arbitrary non-trivial elements of G . Consider the infinite subset $\{x^n y \mid n \in \mathbb{N}\}$. Since $G \in \mathcal{E}_k(\infty)$ then there exist two distinct positive integers i, j such that $[x^i y, x^j y] = 1$. Then by Lemma 1 (parts (1), (2) and (i)) $1 = [[x^i, y],_{k-1} x^j y] [[y, x^j],_{k-1} x^j y] = [[x, y],_{k-1} x^j y]^{i-j}$, and so $[[x, y],_{k-1} x^j y] = 1$. Therefore, by Lemma 1 (parts (1), (2), (i) and (ii)), we get

$$(I) \quad \prod_{r=0}^{k-1} \left[[[x, y],_r x],_{k-r-1} y \right]^{((k-1)!/r!(k-r-1)!)j^r} = 1.$$

Put $t_1 := j$, $K(k-1, r) = \left[[[x, y],_r x],_{k-r-1} y \right]$ and consider the infinite subset $\{x^n y \mid n > t_1\}$. Then by arguing as above, there exists a positive integer $t_2 > t_1$ such that

$$(II) \quad \prod_{r=0}^{k-1} K(k-1, r)^{((k-1)!/r!(k-r-1)!)t_2^r} = 1.$$

Suppose that

$$M(r, a_1, \dots, a_s) = \frac{(k-1)!}{r!(k-r-1)!} \sum_{i_1=s-2}^r \sum_{i_2=s-3}^{i_1-1} \dots \sum_{i_{s-1}=0}^{i_{s-2}-1} a_1^{r-i_1-1} a_2^{i_1-i_2-1} a_3^{i_2-i_3-1} \dots a_{s-1}^{i_{s-2}-i_{s-1}-1} a_s^{i_{s-1}},$$

and $N(r, a_1, \dots, a_s) = (a_s - a_{s-1})M(r, a_1, \dots, a_s)$, for all integers $s > 1$, $s-2 \leq r \leq k-1$ and a_1, \dots, a_s . We note that

$$N(r, a_1, \dots, a_{s+1}) = M(r, a_1, \dots, a_{s-1}, a_{s+1}) - M(r, a_1, \dots, a_s),$$

for all integers $s > 1$, $s-2 \leq r \leq k-1$ and a_1, \dots, a_{s+1} .

By (I) and (II), we have $\prod_{r=1}^{k-1} K(k-1, r)^{N(r, t_1, t_2)} = 1$, and since G is torsion-free, $\prod_{r=1}^{k-1} K(k-1, r)^{M(r, t_1, t_2)} = 1$. We note that from arguing as before, there exists an integer $t_3 > t_2$ such that $\prod_{r=1}^{k-1} K(k-1, r)^{M(r, t_1, t_3)} = 1$. Now suppose, inductively, that there exists a sequence $t_1 < t_2 < t_3 < \dots < t_{s-1} < t_s$ of positive integers such that

$$(*) \quad \prod_{r=s}^{k-1} K(k-1, r)^{M(r, t_1, \dots, t_s)} = 1.$$

Also, there exists an integer $t_{s+1} > t_s$ such that

$$(**) \quad \prod_{r=s}^{k-1} K(k-1, r)^{M(r, t_1, \dots, t_{s-1}, t_{s+1})} = 1.$$

By (*) and (**), we have $\prod_{r=s}^{k-1} K(k-1, r)^{N(r, t_1, \dots, t_{s+1})} = 1$. Since $N(r, t_1, \dots, t_{s+1})$ has a factor of the form $t_{s+1} - t_s$ and G is torsion-free, $\prod_{r=s+1}^{k-1} K(k-1, r)^{M(r, t_1, \dots, t_{s+1})} = 1$. Therefore, we have a sequence $t_1 < t_2 < \dots < t_{k+1}$ of positive integers such that

$$\prod_{r=k-1}^{k-1} K(k-1, r)^{N(r, t_1, \dots, t_{k+1})} = K(k-1, k-1)^{N(k-1, t_1, \dots, t_{k+1})} = 1.$$

But $N(k-1, t_1, \dots, t_{k+1}) = t_{k+1} - t_k > 0$ and $K(k-1, k-1) = [x, y]_{k-1} x = [y, k x]^{-1}$. Hence $[y, k x]^{-(t_{k+1} - t_k)} = 1$ and so $[y, k x] = 1$. This completes the proof. \square

PROOF OF THEOREM 2: If $G/Z_k(G)$ is finite then G is in (\mathcal{N}_k, ∞) and so G belongs to $\mathcal{E}_k(\infty)$. Conversely, by [14, Theorem 1], there exists a finite normal subgroup H of G such that G/H is nilpotent. Let T/H be the torsion subgroup of G/H ; then T is finite and G/T is a torsion-free nilpotent metabelian group. Thus by Lemma 3, $\gamma_{k+1}(G) \leq T$ and so $\gamma_k(G)$ is finite. Hence $G/Z_k(G)$ is also finite [11]. \square

To prove Theorem 3, we need the following key lemma, whose proof is similar to [15, Proposition 5].

LEMMA 4. *Every torsion-free nilpotent $\mathcal{E}_k(\infty)$ -group has nilpotent class bounded by a function of k .*

PROOF: Suppose that G is a torsion-free nilpotent $\mathcal{E}_k(\infty)$ -group. Let G be nilpotent of class c . Then $\gamma_{[c/2]}(G)$ is Abelian, where $[c/2]$ equals $(c+2)/2$ if c is even and $(c+1)/2$ if c is odd. Let A denote the isolator of $\gamma_{[c/2]}(G)$ in G . Then A is also Abelian since G is torsion-free. For any $1 \neq x \in A$ and $y \in G$, consider the infinite subset $\{xy, x^2y, x^3y, \dots\}$. Since $G \in \mathcal{E}_k(\infty)$, there exists two distinct positive integers i, j such that $[x^i y, k x^j y] = 1$. Since A is a normal Abelian subgroup of G , we have

$$1 = [x^i y, k x^j y] = [x^i, k y] [[y, x^j]_{k-1} y] = [x, k y]^i [x, k y]^{-j} = [x, k y]^{i-j}.$$

Therefore $[x, k y] = 1$, since G is torsion-free and $i - j \neq 0$. Hence, we have $[A, k y] = 1$. Since G is torsion-free, it follows from a result of Zel'manov (see [20] p. 166) that A lies in $Z_{f(k)}(G)$, where $f(k)$ is a function of k and independent of the number of generators of G . Thus the nilpotency class of G is at most $[c/2] + f(k)$ and hence $c \leq 2(f(k) + 1)$. \square

PROOF OF THEOREM 3: By [14, Theorem 1], there exists a finite normal subgroup H of G such that G/H is a torsion-free nilpotent group. Thus by Lemma 4, there exists a positive integer t depending only on k such that $\gamma_{t+1}(G) \leq H$ and so $\gamma_{t+1}(G)$ is finite. Hence $G/Z_t(G)$ is also finite [11]. \square

Following [12], we say that a group G is restrained if $\langle x \rangle^{(y)} = \langle x^{y^i} \mid i \in \mathbb{Z} \rangle$ is finitely generated for all x, y in G .

REMARK 1. Note that an \mathcal{E}_k^* -group G with infinite centre Z is k -Engel. For, consider the infinite subsets xZ, yZ for any $x, y \in G$. There exist $z, t \in Z$ such that $[xz, {}_k yt] = 1$ and so $[x, {}_k y] = 1$.

LEMMA 5. Every \mathcal{E}_k^* -group is restrained.

PROOF: Let G be an \mathcal{E}_k^* -group and x, y in G . We must show that $H = \langle x \rangle^{(y)}$ is finitely generated. Assume that y is of infinite order. Consider the two subsets $X = \{x^{y^n} \mid n \in \mathbb{N}\}$ and $Y = \{y^m \mid m \in \mathbb{N}\}$. If X is finite then the centre of $K := \langle x, y \rangle$ is infinite and so by Remark 1, K is k -Engel. Therefore, by [12, Lemma 1(i)], H is finitely generated. Thus, we may assume that X is infinite. Since $G \in \mathcal{E}_k^*$, there exist $n, m \in \mathbb{N}$ such that $[x^{y^n}, {}_k y^m] = 1$ and so $[x, {}_k y^m] = 1$. Thus, arguing as in [12, Lemma 1(i)], $\langle x \rangle^{(y^m)}$ is finitely generated. Therefore $H = \langle x^{y^i} : |i| \leq km \rangle$. This completes the proof. \square

REMARK 2. We note that by [18, Remark 1.2], every infinite residually finite \mathcal{E}_k^* -group is k -Engel.

We are now ready to prove Theorem 4.

PROOF OF THEOREM 4: Let G be an infinite locally graded \mathcal{E}_k^* -group and suppose that $x, y \in G$. We must prove that $[x, {}_k y] = 1$. Assume that there exists an infinite finitely generated subgroup H of G which contains x, y . Let R be the finite residual of H . Then H/R is a finitely generated residually finite group in \mathcal{E}_k^* and so, by Remark 2, H/R is k -Engel. Thus by a theorem of Wilson (see [19, Theorem 2]) H/R is nilpotent. By Lemma 5, H is restrained, therefore by repeated application of [12, Lemma 3], R is finitely generated. If R is finite then H is residually finite and so is k -Engel. Suppose, for a contradiction, that R is infinite. Since G is locally graded, R has a normal proper subgroup of finite index in R , so the finite residual subgroup T of R is proper in R . Therefore R/T is residually finite k -Engel group and so H/T is nilpotent-by-finite. Thus H/T is residually finite and $R \subseteq T$, a contradiction.

We may assume that every finitely generated subgroup of G containing x, y is finite. Thus there exists an infinite locally finite subgroup L which contains x, y and so by [18, Theorem B], L is k -Engel. Therefore in any case, $[x, {}_k y] = 1$ and this completes the first part of Theorem 4. By a result of Kim and Rhemtulla (see [12, Corollary 6]) which asserts that every locally graded bounded Engel group is locally nilpotent, G is locally nilpotent. \square

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