Bulletin of the Iranian Mathematical Society
Vol. 26, No. 1, pp 73-78 (2000)

INFINITE GROUPS WITH AN ENGEL CONDITION ON INFINITE SUBSETS

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Abstract: In this note, we study infinite groups satisfying the condition $\mathcal{E}_2(\infty)$ i.e. groups in which every infinite subset contains two distinct elements x, y such that $[x, y, y] = 1$.

Introduction

Paul Erdös posed the following question:
Let G be an infinite group. If there is no infinite subset of G whose elements do not mutually commute, is there then a finite bound on the cardinality of each such set of elements?
The affirmative answer to this question was obtained by B. H. Neumann. He proved in [9] that a group is centre-by-finite if and only if every infinite subset of the group contains two different commuting elements.
Further questions of a similar nature, with slightly different aspects, have been studied in [6]. These questions are in the following forms:
Let $\mathcal{X}$ be a class of groups and n be a positive integer. we say that a group G is in $(\mathcal{X}, \infty)$ ($(\mathcal{X}, n)$, respectively) if and only if every subset X of G such that $|X| = \infty$ ($|X| = n + 1$, respectively) contains a pair x, y of different elements

MSC: 20F45, 20F99.
Keywords: Engel condition, infinite groups.

generating a subgroup $\langle x, y\rangle$ in $\mathcal{X}$.
1) For which class of groups $\mathcal{X}$ every infinite group in $(\mathcal{X}, \infty)$ belong to the class $(\mathcal{X}, n)$ for some n?
2) How to recognize $(\mathcal{X}, \infty)$ when $\mathcal{X}$ is one or other well-known class of groups?
Therefore, Neumann's Theorem in [9], when $\mathcal{X}$ is the class of abelian groups, gives a positive answer to the first question and also gives rise to a sensible answer in connection with the second one. Many people have worked on the above questions.

Let $\mathcal{N}$ and $\mathcal{N}_k$ be the classes of nilpotent and nilpotent groups of class at most k, respectively. For a group G, we denote by $Z_n(G)$ and $\Gamma_n(G)$, respectively, the $(n+1)$-th term of the upper central series of G and the n-th term of the lower central series of G.
In [6] Lennox and Wiegold studied the class $(\mathcal{N}, \infty)$ and proved that a finitely generated soluble group is in $(\mathcal{N}, \infty)$ if and only if it is finite-by-nilpotent.
In [2] Abdollahi and Taeri, proved that if G is a finitely generated soluble group, then $G \in (\mathcal{N}_k, \infty)$ if and only if G is finite by a group in which every two generator subgroup is nilpotent of class at most k. Also, Corollary 3 of [2] gives a positive answer to the first question in the class of finitely generated soluble groups when $\mathcal{X}$ is $\mathcal{N}_k$.
In [7] Longobardi and Maj introduced the classes $\mathcal{E}(\infty)$ and $\mathcal{E}_k(\infty)$ as follows: A group $G \in \mathcal{E}(\infty)$ if and only if every infinite subset X of G, contains different elements x, y such that $[x,_k y] = 1$ for some integer $k = k(x,y) \geq 1$. If the integer k is the same for all infinite subsets of G, we say that G is in the class $\mathcal{E}_k(\infty)$. It is easy to see that all the classes defined above are closed with respect to forming subgroups and homomorphic images and $\mathcal{E}_k(\infty) \subseteq \mathcal{E}(\infty)$, $(\mathcal{N}_k, \infty) \subseteq (\mathcal{N}, \infty)$. Clearly every $(\mathcal{N}_k, \infty)$-group is also a $\mathcal{E}_k(\infty)$-group.
It is proved in [7] that a finitely generated soluble group is in $\mathcal{E}(\infty)$ if and only if it is finite-by-nilpotent. This result is a generalization of Theorem A of [6]. Moreover, it is proved that a finitely generated soluble group G is in $\mathcal{E}_2(\infty)$ if and only if $G/R(G)$ is finite, where $R(G)$ is the characteristic subgroup of G consisting of all right 2-Engel elements of G.
In [3] Delizia proved that a finitely generated soluble group G is in $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite. In [1] Abdollahi proved that a finitely generated soluble group G is in $\mathcal{E}_2(\infty)$ if and only if $G/Z_2(G)$ is finite. Since $Z_2(G) \subseteq R(G)$, the latter result improves Theorem 2 of [7], also it improves Theorem A

of Delizia [3], since it shows that a finitely generated soluble group in $\mathcal{E}_2(\infty)$ is in $(\mathcal{N}_2, \infty)$.
Delizia in [4], proved that a finitely generated residually finite group G is in $(\mathcal{N}_2, \infty)$ if and only if $G/Z_2(G)$ is finite.
Let us recall that a group G is locally graded if every finitely generated non-trivial subgroup of G has a non-trivial finite image. More recently in [5], Delizia, Rhemtulla and Smith prove that if G is a finitely generated locally graded $(\mathcal{N}_k, \infty)$-group, then there is a positive integer c depending only on k such that $G/Z_c(G)$ is finite. Theorem of [5] is essentially a generalization of the results in [2], [3], [4] and [6].

In this note, we investigate finitely generated residually finite groups in the class $\mathcal{E}_2(\infty)$. We conjecture that the result of [4] is valid even if we replace $(\mathcal{N}_2, \infty)$ by $\mathcal{E}_2(\infty)$, that is:

Conjecture. *Let G be a finitely generated residually finite group. Then $G \in \mathcal{E}_2(\infty)$ if and only if $G/Z_2(G)$ is finite.*

The Conjecture is true if we add the additional condition that G is of finite rank. In fact our main result is:

Theorem. *Let G be an infinite finitely generated residually finite $\mathcal{E}_2(\infty)$-group of finite rank. Then $G/Z_2(G)$ is finite.*

Recall that a group is of finite rank if and only if there is a positive integer d such that every finitely generated subgroup of G can be generated by d elements (see page 34 of [10]).

Proofs

We need the following lemma in the proof of the Theorem.

Lemma 1. *Every finitely generated nilpotent-by-finite $\mathcal{E}_k(\infty)$-group is finite-by-nilpotent.*

Proof. Let G be a finitely generated nilpotent-by-finite $\mathcal{E}_k(\infty)$-group. Suppose, for a contradiction, that $G \cong G/1$ is not finite-by-nilpotent. Since G is finitely generated nilpotent-by-finite, G satisfies the maximal condition on subgroups. Therefore there exists among the normal subgroups of G one, N say, which is maximal subject to G/N is not finite-by-nilpotent. Now the group

$\overline{G} = G/N$ is a finitely generated nilpotent-by-finite $\mathcal{E}_k(\infty)$-group, and every proper quotient group of $\overline{G}$ is finite-by-nilpotent. Since $\overline{G}$ is not finite-by-nilpotent, $\overline{G}$ is not finite. There exists a normal nilpotent subgroup H of finite index in $\overline{G}$, since $\overline{G}$ is nilpotent-by-finite. Thus H is an infinite finitely generated nilpotent group and so H has an infinite free abelian characteristic subgroup A. Now we claim that A is a subset of the set of all right Engel elements of $\overline{G}$, $R(\overline{G})$. Suppose that the claim is true, then A is a subgroup of the hypercentre of G by Theorem 7.21 of [11], so that in particular the centre $Z(\overline{G})$ is non-trivial. Thus $\overline{G}/Z(\overline{G})$ is finite-by-nilpotent. Now Theorem 4.25 of [10] yields that $\overline{G}$ is finite-by-nilpotent, a contradiction.
Now we prove the claim. Let $1 \neq y$ and x be any elements of A and $\overline{G}$ respectively, and consider the infinite sequence

$$yx, y^2x, \ldots, y^rx, \ldots.$$

By hypothesis there exist two different positive integers m, n such that the repeated commutator

$$[y^nx,_k y^mx] = 1.$$

Since A is a normal abelian subgroup of $\overline{G}$, $[y,_k x]^{n-m} = 1$. Thus $[y,_k x] = 1$, since A is torsion-free. Hence $y \in R(\overline{G})$. □

Now we are ready to prove the Theorem.

Proof of the Theorem. By Corollary 2 of [8], G is soluble-by-finite. Then there is a normal subgroup H of G such that H is soluble and G/H is finite, so H is also finitely generated. $H/Z_2(H)$ is finite by [1] and so is $G/Z_2(H)$. Since $Z_2(H)$ is nilpotent , G is a finitely generated nilpotent-by-finite group. Therefore G is finite-by-nilpotent, by Lemma 1. Thus there exists a finite normal subgroup K of G such that G/K is nilpotent. Since $G/K \in \mathcal{E}_2(\infty)$ and G/K is finitely generated nilpotent, by [1], $\frac{(G/K)}{Z_2(G/K)}$ is finite. Then by Corollary 2 to Theorem 4.21 of [10], $\Gamma_3(G/K) = \frac{\Gamma_3(G)K}{K}$ is finite, so $\Gamma_3(G)$ is finite. Now it follows from Theorem 4.24 of [10], that $G/Z_2(G)$ is finite. □

We conclude this note by proving the following results on infinite $\mathcal{E}_k(\infty)$-groups.

Lemma 2. *Let G be an infinite $\mathcal{E}_k(\infty)$-group. Then the centralizer $C_G(x)$ of every element x in G is infinite.*

Proof. Suppose, for a contradiction, that there exists x in G such that $|C_G(x)| = n$ is finite. Then the set of conjugates $X = \{x^g \mid g \in G\}$ is infinite. List the elements of X as $x_1, x_2, \ldots$ under some well ordering $\leq$ so that $x_i < x_j$ if $i < j$. Consider the set $X^{(2)}$ of all 2-element subsets of X. For each $s \in X^{(2)}$, list the elements x_{i_1}, x_{i_2} of s in ascending order given by $\leq$ and write $\hat{s} = (x_{i_1}, x_{i_2})$. Create 3 sets, one U_σ for each permutation σ on the set $\{1, 2\}$ and V as follows: for each $s \in X^{(2)}$, $\hat{s} = (x_{i_1}, x_{i_2})$, put $s \in U_\sigma$ if $[x_{i_{\sigma(1)}},_k x_{i_{\sigma(2)}}] = 1$ and put $s \in V$ if $s \notin U_\sigma$, for any σ. By Ramsey's Theorem, there exists an infinite subset $X_0 \subseteq X$ such that $X_0^{(2)} \subseteq U_\sigma$, for some σ or $X_0^{(2)} \subseteq V$. Suppose, for a contradiction, that $X_0^{(2)} \subseteq V$. By the property $\mathcal{E}_k(\infty)$, there exist x_i, x_j in X_0 such that $[x_i,_k x_j] = 1$, but then $s = \{x_i, x_j\}$ lies in some U_σ, a contradiction. Thus $X_0^{(2)} \subseteq U_\sigma$ for some σ. Moreover, by restricting the order $\leq$ to X_0, we may assume that $X_0 = \{x_1, x_2, \ldots\}$ and $x_i < x_j$ if $i < j$. Hence for any $i_1 < i_2$, $[x_{i_{\sigma(1)}},_k x_{i_{\sigma(2)}}] = 1$. Consider the set

$$T_v = \{[x_{i_{\sigma(1)}},_{k-1} x_{i_{\sigma(2)}}] \mid i_1 < i_2 \text{ and } i_{\sigma(2)} = v\}.$$

If $|T_v| > n$ for some value of v, then it follows that the centralizer of x_v has order greater than n, a contradiction, since x_v is a conjugate of x and $|C_G(x)| = n$. Now let l be the least integer for which a bound N exists such that for all choices of integer v, $\{[x_{i_{\sigma(1)}},_l x_{i_{\sigma(2)}}] \mid i_1 < i_2 \text{ and } i_{\sigma(2)} = v\}$ has order at most N. Then $l \leq k-1$ as we have seen above. Also $l \geq 2$, since X_0 is infinite. Then we have the situation where there exist integers v and $N > 1$ such that $N \geq |\{[x_{i_{\sigma(1)}},_l x_{i_{\sigma(2)}}] \mid i_1 < i_2 \text{ and } i_{\sigma(2)} = v\}|$ and

$$N^t \leq |\{[x_{i_{\sigma(1)}},_{l-1} x_{i_{\sigma(2)}}] \mid i_1 < i_2 \text{ and } i_{\sigma(2)} = v\}|,$$

for a suitable t such that $N^t > n$. It follows that $|C_G(x_v)| \geq N^t > n$, a contradiction. This completes the proof. □

From Lemma 2 it follows, with straightforward arguments:

Corollary 3. *Let G be an infinite $\mathcal{E}_k(\infty)$-group. Then every element x in G is contained in an infinite abelian subgroup of G.*

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