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A characterization of infinite 3-abelian groups

By

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Abstract. In this note we prove that every infinite group G is 3-abelian (i.e. $(ab)^3 = a^3b^3$ for all a, b in G) if and only if in every two infinite subsets X and Y of G there exist $x \in X$ and $y \in Y$ such that $(xy)^3 = x^3y^3$.

Introduction. In response to a question of P. Erdös, B. H. Neumann showed that an infinite group G is centre-by-finite if and only if every infinite subset of G contains two distinct commuting elements [8]. Since this first paper, problems of a similar nature have been the object of several articles (for example [2], [3], [6], [10]).

Let w be a word in the free group of rank $n > 0$. For $\mathscr{V} = \mathscr{V}(w)$ the variety of groups defined by the law $w(x_1, \dots, x_n) = 1$, P. Longobardi, M. Maj and A. H. Rhemtulla in [7] defined $\mathscr{V}^* = \mathscr{V}(w^*)$ to be the class of all groups G in which for any infinite subsets $X_1, \dots, X_n$ there exist $x_i \in X_i$, $1 \leqq i \leqq n$, such that $w(x_1, \dots, x_n) = 1$ and raised the question of whether $\mathscr{F} \cup \mathscr{V}(w) = \mathscr{V}(w^*)$ is true; $\mathscr{F}$ being the class of finite groups.

There is no example, so far, of an infinite group in $\mathscr{V}(w^*) \backslash \mathscr{V}(w)$. In considering the question many authors have obtained the equality for certain words (see [1], [5], [7], [12], and [13]) and for certain class of groups (see [1]).

Here we add another word for which the equality holds: Let $\mathscr{A}_3$ be the variety of 3-abelian groups; namely the variety defined by the law $(xy)^3(x^3y^3)^{-1} = 1$, then we prove

Theorem. *Every infinite $\mathscr{A}_3^*$-group is 3-abelian.*

The variety $\mathscr{A}_3$ has been studied by F. W. Levi who has shown in [4] that a group G is 3-abelian if and only if G is a 2-Engel group and the derived subgroup G' of G has exponent dividing 3.

In order to conclude our characterization, we need the following lemmas. If G is a group and a, x and y are arbitrary elements of G then, as usual, we shall denote by $Z(G), C_G(x)$ and a^x, the centre of G, the centralizer of x in G and the conjugate of a by x, respectively and also $[a, x] = a^{-1}a^x$ and $[a, x, y] = [[a, x], y]$.

Lemma 1. *Let G be a group and let x and y be any two elements in G.*

(i) *If $(xy)^3 = x^3y^3$ and $(xy^2)^3 = x^3y^6$ then $[x, y, y] = 1$.*
(ii) *If $(xy)^3 = x^3y^3$, $(yx)^3 = y^3x^3$ and $(x^2y)^3 = x^6y^3$ then $[y^{-1}, x, x] = 1$.*
(iii) *If $(xy)^3 = x^3y^3$ and $(x^{-1}y)^3 = x^{-3}y^3$ then $[x, y^{-1}] \in C_G(x^2)$.*
(iv) *If $(xy)^3 = x^3y^3$, $[x, y, y] = 1$ and $[y, x, x] = 1$ then $[x^3, y] = 1$.*

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Proof. It is easy to show that for any two elements x and y in a group

$$(xy)^3 = x^3y^3 \text{ if and only if } [x, y^{-1}][y^{-2}, x^{-1}] = 1$$

from which all the statements of the lemma will follow. $\square$

Lemma 2. *Let G be an infinite $\mathscr{A}_3^*$-group and suppose that a is an element of G such that $C_G(a^2)$ is infinite. Then $C_G(a)$ is also infinite.*

Proof. Suppose, for a contradiction, that $C_G(a)$ is finite. Then the set $X = \{a^g \mid g \in C_G(a^2)\}$ is infinite. Consider the sets $\mathfrak{X}_1 = \{\{x, y\} \subset X \mid (xy)^2 = a^4\}$ and $\mathfrak{X}_2 = X^{(2)} \backslash \mathfrak{X}_1$, where $X^{(2)}$ is the set of all 2-element subsets of X, note that for any $x, y \in X$ we have $(xy)^2 = a^4 \Leftrightarrow (yx)^2 = a^4$. Clearly $X^{(2)}=\mathfrak{X}_1 \cup \mathfrak{X}_2$. By Ramsey's Theorem [9], there exists an infinite subset X_0 of X such that $X_0^{(2)} \subseteqq \mathfrak{X}_1$ or $X_0^{(2)} \subseteqq \mathfrak{X}_2$. But if $X_0^{(2)} \subseteqq \mathfrak{X}_2$, then partition X_0 into two infinite subsets Y_1 and Y_2. By the property $\mathscr{A}_3^*$, there exist $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $(y_1y_2)^3 = y_1^3y_2^3$ which yields $(y_2y_1)^2 = y_1^2y_2^2$. But $t^2 = a^2$ for all $t \in X$ therefore $(y_2y_1)^2 = a^4$. Hence $\{y_1, y_2\}$ lies in $\mathfrak{X}_1$, a contradiction. Thus $X_0^{(2)} \subseteqq \mathfrak{X}_1$, and so for all $x, y \in X_0$ we have $(xy)^2 = a^4 = a^2a^2 = x^2y^2$. Therefore $xy = yx$ for all $x, y \in X_0$ and so $X_0 \subseteqq C_G(x)$ for all $x \in X_0$. Thus $C_G(x)$ is infinite for all $x \in X_0$ hence $C_G(a)$ is infinite, which is a contradiction. $\square$

The following lemma is the key to the proof of the theorem.

Lemma 3. *Let G be an infinite $\mathscr{A}_3^*$-group. Then $C_G(a)$ is infinite for all a in G.*

Proof. Suppose, for a contradiction, that $C_G(a)$ is finite for some a in G, thus $X = \{a^g \mid g \in G\}$ is infinite. Now, list the elements of X as $x_1, x_2, \ldots$ under some well order $\leqq$ so that $x_i < x_j$ if $i < j$. Consider the set $X^{(2)}$ of all 2-element subsets of X. For each $s \in X^{(2)}$, list the elements x_{i_1}, x_{i_2} of s in ascending order given by $\leqq$ and write $\tilde{s} = (x_{i_1}, x_{i_2})$. Create 3 sets. One U_σ for each permutation σ of $\{1, 2\}$ and V. For each $s \in X^{(2)}$, $\tilde{s} = (x_{i_1}, x_{i_2})$, put $s \in U_\sigma$ if $(x_{i_{\sigma(1)}}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^3x_{i_{\sigma(2)}}^3$ and put s in V if $s \notin U_\sigma$ for any σ. By Ramsey's Theorem [9], there exists an infinite subset $X_0 \subseteqq X$ such that $X_0^{(2)} \subseteqq U_\sigma$ for some σ or $X_0^{(2)} \subseteqq V$. Suppose, if possible, that $X_0^{(2)} \subseteqq V$, then partitioning the set X_0 into two infinite subsets Y_1 and Y_2, the property $\mathscr{A}_3^*$, yields elements $y_1 \in Y_1$ and $y_2 \in Y_2$ such that $(y_1y_2)^3 = y_1^3y_2^3$; but then $s = \{y_1, y_2\}$ lies in some U_σ and not in V, a contradiction.

We may thus assume that $X_0^{(2)} \subseteqq U_\sigma$ for some permutation σ. Moreover, by restricting the order $\leqq$ to X_0, we may assume that $X_0 = \{x_1, x_2, \ldots\}$ and $x_i < x_j$ if $i < j$. Hence for any $i_1 < i_2$, $(x_{i_{\sigma(1)}}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^3x_{i_{\sigma(2)}}^3$. Now create again 3 sets. One T_τ for each permutation τ of $\{1, 2\}$ and W. For each $s \in X_0^{(2)}$, $\tilde{s} = (x_{i_1}, x_{i_2})$, put $s \in T_\tau$ if $(x_{i_{\tau(1)}}^{-1}x_{i_{\tau(2)}})^3 = x_{i_{\tau(1)}}^{-3}x_{i_{\tau(2)}}^3$ and put s in W if $s \notin T_\tau$ for any τ. By Ramsey's Theorem, there exists an infinite subset $X_1 \subseteqq X_0$ such that $X_1^{(2)} \subseteqq T_\tau$ or $X_1^{(2)} \subseteqq W$. Suppose, if possible, that $X_1^{(2)} \subseteqq W$, then partition the set X_1 into two infinite subsets Z_1 and Z_2. Consider infinite subsets $Z_1^{-1} = \{z^{-1} \mid z \in Z_1\}$ and Z_2. By the property $\mathscr{A}_3^*$, there exist $z_1 \in Z_1$ and $z_2 \in Z_2$ such that $(z_1^{-1}z_2)^3 = z_1^{-3}z_2^3$; but then $s = \{z_1, z_2\}$ lies in some T_τ and not in W, a contradiction.

We may thus assume that $X_1^{(2)} \subseteqq T_\tau$ for some permutation τ. Moreover, by restricting the order $\leqq$ to X_1 we may assume that $X_1 = \{x_1, x_2, \ldots\}$ and $x_i < x_j$ if $i < j$. Hence for any $i_1 < i_2$ $(x_{i_{\sigma(1)}}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^3x_{i_{\sigma(2)}}^3$ and $(x_{i_{\tau(1)}}^{-1}x_{i_{\tau(2)}})^3 = x_{i_{\tau(1)}}^{-3}x_{i_{\tau(2)}}^3$. If $\sigma \neq \tau$ then $\sigma(1) = \tau(2)$ and

$\sigma(2)=\tau(1)$. Thus $(x_{i_{\sigma(2)}}^{-1}x_{i_{\sigma(1)}})^3 = x_{i_{\sigma(2)}}^{-3}x_{i_{\sigma(1)}}^3$ for any $i_1<i_2$. By inversing two sides of the latter relation, we obtain $(x_{i_{\sigma(1)}}^{-1}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^{-3}x_{i_{\sigma(2)}}^3$. Therefore $(x_{i_{\sigma(1)}}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^3x_{i_{\sigma(2)}}^3$ and $(x_{i_{\sigma(1)}}^{-1}x_{i_{\sigma(2)}})^3 = x_{i_{\sigma(1)}}^{-3}x_{i_{\sigma(2)}}^3$ for any $i_1<i_2$. Now, by Lemma 1(iii), $[x_{i_{\sigma(1)}}, x_{i_{\sigma(2)}}^{-1}] \in C_G(x_{i_{\sigma(1)}}^2)$ for any $i_1<i_2$. Suppose that $x_{i_{\sigma(j)}} = a^{g_{i_{\sigma(j)}}}$ for some $g_{i_{\sigma(j)}} \in G$, $j\in\{1,2\}$. Therefore $[a,(a^{-1})^{g_{i_{\sigma(2)}}g_{i_{\sigma(1)}}^{-1}}] \in C_G(a^2)$ for any $i_1<i_2$. Let $T=\{g_{i_{\sigma(2)}}g_{i_{\sigma(1)}}^{-1} \mid i_1 \text{ is fixed and } i_1<i_2\}$, clearly T is infinite. Suppose that $M=\{[a,(a^{-1})^t] \mid t\in T\}$ is infinite. Then $C_G(a^2)$ is infinite and so by Lemma 2, $C_G(a)$ is infinite, a contradiction. Thus M is finite. Since T is infinite and M is finite, there exist an infinite subset $T_0 \subseteqq T$ and an element $t_0\in T_0$ such that $[a,(a^{-1})^t]=[a,(a^{-1})^{t_0}]$ for all $t\in T_0$. Thus $(a^{-1})^t a^{t_0}\in C_G(a)$ for all $t\in T_0$. Suppose that $N=\{(a^{-1})^t a^{t_0} \mid t\in T_0\}$ is infinite, then $C_G(a)$ is infinite, a contradiction. Thus N is finite and so there exist an infinite subset T_1 of T_0 and $t_1\in T_1$ such that $(a^{-1})^t a^{t_0} = (a^{-1})^{t_1}a^{t_0}$ for all $t\in T_1$, and so $tt_1^{-1}\in C_G(a)$ for all $t\in T_1$. Therefore $C_G(a)$ is infinite, a contradiction. □

Since by Lemma 3, for any infinite subgroup H of an infinite $\mathscr{A}_3^*$-group and any h in H, $C_H(h)$ is infinite, we have

Corollary 4. *Let G be an infinite $\mathscr{A}_3^*$-group. Then for every element x of G there exists an infinite abelian subgroup A of G containing x.*

We note that, by Lemma 3 in [1], every infinite $\mathscr{A}_3^*$-group with infinite centre is 3-abelian.

Lemma 5. *Let G be an infinite $\mathscr{A}_3^*$-group, A be an infinite abelian subgroup of G and $y_1,\dots,y_m\in G$. Then*

$$B=\{a\in A \mid (ay_i)^3(a^3y_i^3)^{-1} = (y_ia)^3(y_i^3a^3)^{-1}=1\ ;\ i=1,\dots,m\}$$

is a cofinite set in A.

Proof. We must show that $A\backslash B$ is finite. We argue by induction on m. Let $m=1$. Consider the set $Y=\{y_1^a \mid a\in A\}$. Suppose that Y is finite, then the index $|A:C_A(y_1)|$ is finite too, hence $C_A(y_1)$ is infinite and contained in the centre of $H=\langle A,y_1\rangle$. This means that $Z(H)$ is infinite, and so H is a 3-abelian group, thus $A=B$. So we may assume, without loss of generality, that Y is infinite. We want to show that $A\backslash B_1$ is finite where $B_1=\{a\in A \mid (ay_1)^3=a^3y_1^3\}$. Suppose not, then since Y is infinite, by the property $\mathscr{A}_3^*$, there are elements $a\in A\backslash B_1$ and $b\in A$ such that $(ay_1^b)^3=a^3(y_1^3)^b$. Since $ab=ba$ we have $a\in B_1$, which is a contradiction. Thus $A\backslash B_1$ is finite. Similarly the set $A\backslash B_2$ is finite where $B_2=\{a\in A \mid (y_1a)^3=y_1^3a^3\}$. Therefore $A\backslash B$ is finite since $B_1\cap B_2=B$. Now suppose, inductively, that $m>1$ and $C=\{a\in A \mid (ay_i)^3(a^3y_i^3)^{-1}=(y_ia)^3(y_i^3a^3)^{-1}=1$ $i=1,\dots,m-1\}$ is a cofinite set in A. As in the case $m=1$, we have that $D=\{a\in A \mid (ay_m)^3(a^3y_m^3)^{-1}=(y_ma)^3(y_m^3a^3)^{-1}=1\}$ is a cofinite set in A. Thus $A\backslash B$ is finite since $B=D\cap C$. Therefore the induction is complete. □

Lemma 6. *Let G be an infinite $\mathscr{A}_3^*$-group. Then every element of order 2 lies in the centre of G.*

Proof. Let y be an element of order 2 and x be an arbitrary element of G. By Corollary 4, there exists an infinite abelian subgroup A of G such that $x\in A$. By Lemma 5, the set $B=\{a\in A \mid (ay)^3=a^3y^3\}$ is a cofinite set in A . Let $b\in B$. Since $(by)^3=b^3y^3$ and $y^2=1$, we have $[b, y] = 1$. Therefore $B \subseteqq C_A(y)$ and so $\langle A, y\rangle$ is an infinite $\mathscr{A}_3^*$-group with infinite centre. Thus $\langle A, y\rangle$ is 3-abelian. Now since $y^2 = 1$ it follows from $(xy)^3 = x^3y^3$ that $xy = yx$. $\square$

Lemma 7. *Let G be an infinite* $\mathscr{A}_3^*$*-group. Then* $[x^3, y] = 1$ *for all* $x, y \in G$.

Proof. By Corollary 4, there exists an infinite abelian subgroup A of G such that $x \in A$. Consider the set B defined as in Lemma 5, with respect to A and the elements $y_1 = y, y_2 = y^{-1}, y_3 = y^2$. By Lemma 5, B is a cofinite subset of A. Set $C = \{b \in B \mid (b^2y^{-1})^3 = b^6y^{-3}\}$. Suppose, if possible, that $A\backslash C$ is infinite, then $D = (A\backslash C) \cap B$ is infinite too. If $D^2 = \{d^2 \mid d \in D\}$ is finite then the set of all elements of order 2 in the abelian group A is infinite. Therefore $Z(G)$ is infinite by Lemma 6. Hence G is 3-abelian and so $C = B$, a contradiction. Thus D^2 is infinite. Also if $Y = \{(y^{-1})^b \mid b \in D\}$ is finite then $Z(\langle A, y\rangle)$ is infinite and so $\langle A, y\rangle$ is 3-abelian. Thus $B = C$, a contradiction. Now consider the infinite subsets D^2 and Y. By the property $\mathscr{A}_3^*$, there exist $b_1, b_2 \in D$ such that $(b_1^2(y^{-1})^{b_2})^3 = b_1^6(y^{-3})^{b_2}$, thus $(b_1^2y^{-1})^3 = b_1^6y^{-3}$ and so $b_1 \in C$, a contradiction. Therefore C is a cofinite set in A. If $b \in C$ then $(by)^3 = b^3y^3$ and $(by^2)^3 = b^3y^6$, by Lemma 1 (i), $[b, y, y] = 1$. Since $(by^{-1})^3 = b^3y^{-3}$, $(y^{-1}b)^3 = y^{-3}b^3$ and $(b^2y^{-1})^3 = b^6y^{-3}$, Lemma 1 (ii) yields $[y, b, b] = 1$. Also since $(by)^3 = b^3y^3$, by Lemma 1 (iv), we have $[b^3, y] = 1$. Consider the infinite set $xC = \{xb \mid b \in C\}$. Since $A\backslash C$ is finite and $xC \subseteqq A$, $xC \cap C$ is non-empty. So there is an element $b \in C$ such that $xb \in C$. Therefore $[(xb)^3, y] = 1$, hence $[x^3, y] = 1$, since $[b^3, y] = [x, b] = 1$. $\square$

Now we are ready to give the proof of the theorem.

Proof of the theorem. Let G be an infinite $\mathscr{A}_3^*$-group. By Lemma 7, $G^3 = \langle x^3 \mid x \in G\rangle$ lies in $Z(G)$. Thus $G/Z(G)$ is a group of exponent dividing 3 and by Theorem 7.14 of [11], $G/Z(G)$ is 2-Engel and by Corollary 3 page 45 in [11], $G/Z(G)$ is nilpotent. Thus G is a nilpotent group and Theorem 1 in [1] gives the result. $\square$

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